

Bell-states diagonal entanglement witnesses for relativistic and non-relativistic multispinor systems in arbitrary dimensions

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Abstract

Two kinds of Bell-states diagonal (BSD) entanglement witnesses (EW) are constructed by using the algebra of Dirac γ matrices in the space-time of arbitrary dimension d , where the first kind can detect some BSD relativistic and non-relativistic m -partite multispinor bound entangled states in Hilbert space of dimension $2^{m\lfloor d/2 \rfloor}$, including the bipartite Bell-type and iso-concurrence type states in the four-dimensional space-time ($d = 4$). By using the connection between Hilbert-Schmidt measure and the optimal EWs associated with states, it is shown that as far as the spin quantum correlations is concerned, the amount of entanglement is not a relativistic scalar and has no invariant meaning. The introduced EWs are manipulated via the linear programming (LP) which can be solved exactly by using simplex method. The decomposability or non-decomposability of these EWs is investigated, where the region of non-decomposable EWs of the first kind is partially determined and it is shown that, all of the EWs of the second kind are decomposable. These EWs have the preference that in the bipartite systems, they can determine the region of separable states, i.e., bipartite non-detectable density matrices of the same type as the EWs of the first kind are necessarily separable. Also, multispinor EWs with non-polygon feasible regions are provided, where the problem is solved by approximate LP, and in contrary to the exactly manipulatable EWs, both the first and second kind of the optimal approximate EWs can detect some bound entangled states.

Keywords: Relativistic entanglement, Entanglement Witness, Multispinor, Linear Programming, Feasible Region.

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1 Introduction

Entanglement is one of the most fascinating features of quantum mechanics and a lot of work has been devoted to this topic in the recent years [1]-[13]. It has recently been recognized that entanglement is a very important resource in quantum information processing [14] such as teleportation [15] and clock synchronization [16]. On the other hand, there is a natural interest in studying nonlocal quantum correlations in the framework of special relativity [17]. Relativistic quantum information processing is of growing interest not only for the logical completeness but also with regard to new features, such as the physical bounds on information transfer, processing and the errors provided by the full relativistic treatments (see the review [18]). Tracing back to Bell's famous re-imagining of the Einstein-Podolsky-Rosen paradox [19], a standard system of interest is two particles with spins entangled due to their production in the decay or scattering. Various authors have considered the entanglement of two relativistic particles [20]-[37]. Some of these papers discuss the covariance of the Bell's inequality and show that the violation of this inequality decreases with increasing the velocity of the moving frame. Although the results of this type produce interesting insights to the relativistic quantum information, but it should be noticed that decreasing the amount of violation of the Bell inequality do not imply that the amount of entanglement decreases under the Lorentz transformation, since the violation of Bell inequalities are tools only for detection of non-locality and can not be considered as a suitable entanglement measure. On the other hand, these papers have studied only pure relativistic states where, the entanglement between spins of two electrons is considered. In this paper, we take the approach of so-called entanglement witnesses (EW's) [2] to distinguish separable mixed states from entangled ones (an EW for a given entangled state ρ is an observable W whose expectation value is non-negative on any separable state, but strictly negative on the entangled state ρ) and by constructing EWs called Bell-states diagonal (BSD) multispinor EWs, present a general scheme which can be used for

studying the entanglement properties of relativistic and non-relativistic multispinor systems in an arbitrary space-time dimension d . It should be noticed that, the framework of Bell inequalities fits in the scheme of EWs such that as it has been discussed in Ref. [38], each Bell inequality can be viewed as a particular example of an entanglement witness. In fact, the Bell inequalities are corresponded to non-optimal EWs and can be only used as criteria for detection of entanglement. Despite of the fact that the EWs are designed mainly for detection of the entanglement, it has been shown [39] that the optimal EW associated with a density matrix ρ -in the sense that, the expectation value of the optimal EW (associated with ρ) over ρ is the most negative value between the expectation values of other EWs over ρ - can be used as measure of entanglement quantifying the amount of entanglement of ρ . In Refs. [40], [41] a connection between Hilbert-Schmidt measure and the optimal EW associated with a state has been discussed. We will use this connection in order to show that, the amount of entanglement between the spins of electron and positron (for a given momentum \vec{p}) in a bipartite system with space-time dimension $d = 4$ is not Lorentz invariant, where this result is in agreement with those of Ref. [21]. There has been much work on the separability problem, particularly from the Innsbruck-Hannover group, as reviewed in [3, 12], which emphasizes convexity and proceeds by characterizing EWs in terms of their extreme points, the so-called optimal EWs [4], and PPT entangled states (those density matrices which have positive partial transposition with respect to each subsystem [42]) in terms of their extreme points, the edge PPT entangled states [5, 7]. In fact, in order to a hermitian operator W be an EW, it must possess at least one negative eigenvalue and the expectation value of W over any separable state must be non-negative. Therefore, for determination of EWs, one needs to determine the minimum value of this expectation value over the feasible region (the minimum value must be non-negative) and hence the problem reduces to an optimization over the convex set of feasible region. For example, in [43, 44] the manipulation of generic Bell-state diagonal EWs has been reduced to such an optimization problem. It has been shown that, if the feasible region for this optimization

constructs a polygon by itself, the corresponding boundary points of the convex hull will minimize exactly the optimization problem. This problem is called linear programming (LP) and the simplex method is the easiest way of solving it [45]. If the feasible region is not a polygon, with the help of tangent planes in this region at points which are determined either analytically or numerically, one can define a new convex hull which is a polygon and has encircled the feasible region. The points on the boundary of the polygon can approximately determine the minimum value of the optimization problem. Thus the approximated value is obtained via LP. In general, it is difficult to find this region and solve the corresponding optimization problem; thus, it is difficult to find any generic multipartite EW. Recently, in Ref. [46], a new class of EWs called reduction type EWs has been introduced for which the feasible regions turn out to be convex polygons. In this work, we construct two kinds of BSD multispinor EWs by using the algebra of Dirac γ matrices in the space-time of arbitrary dimension d , where the first kind can detect some m -partite BSD non-relativistic multispinor PPT entangled states with Hilbert space of dimension $2^{m\lfloor d/2 \rfloor}$. Furthermore, in the four-dimensional space-time ($d = 4$), we introduce 16 Bell-type and iso-concurrence type states (for definition and entanglement properties of the iso-concurrence states, the reader is referred to [8]-[11]) and show that, these states (including the spinor “EPR” state [47] which is a special kind of iso-concurrence type entangled states) are detected by the constructed multispinor EWs. Moreover, by using the bipartite optimal EWs of the first kind and the Hilbert-Schmidt measure of entanglement, we calculate the amount of entanglement for some kinds of BSD density matrices (in the four-dimensional space-time) in the rest frame and the corresponding Lorentz transformed states, where the result shows that the spin entanglement of these states (for a given momentum \vec{p}) is not relativistic invariant. By using the prescriptions of References [43], [44], the introduced EWs can be manipulated via the LP which can be solved exactly via simplex method. The region of entangled states which can be detected via each kind of EWs is determined. It is shown that, bipartite non-detectable density matrices of the same type (their structures are

the same except for the positivity of density matrices) as the EWs of the first kind are necessarily separable. Also, we discuss the decomposability or non-decomposability of the EWs, where the region of non-decomposable EWs of the first kind is partially determined and it is shown that, all of the EWs of the second kind are decomposable. It should be noticed that, without using the techniques such as LP optimization method construction of optimal EWs specially non-decomposable ones is not an easy task and as far as we know, this work is a first step toward a relativistic extension of quantum entanglement in multispinor systems with mixed states specially PPT mixed ones. Moreover, similar to the References [43], [44] and [46], one can obtain some decomposable or non-decomposable positive maps from the introduced multispinor EWs by using the Jamiołkowski isomorphism [5], [6] but this is not treated in this work. We discuss also examples of EWs (in each kind) for which the feasible regions are not polygon and so, the region of EWs can be approximately determined by LP (in these cases, the convex optimization is reduced to the LP one). It is shown that, in contrary to the exactly manipulatable EWs, both the first and second kind of the optimal approximate EWs can detect some PPT entangled states.

The paper is organized as follows: In section 2, some of the definitions and properties related to the EWs, linear programming (LP) and general scheme for manipulation of EWs by using the exact and approximate LP method are reviewed. In section 3, two kinds of BSD multispinor EWs in space-time with arbitrary dimension are introduced. Also, the optimality of some of these EWs in each kind is proved. Section 4 is devoted to the region of entangled states which can be detected by the introduced EWs. In particular, in the bipartite systems in four-dimensional space-time, the Bell-type and iso-concurrence type entangled states are defined and it is shown that the amount of spin entanglement measured by the Hilbert-Schmidt measure is not Lorentz invariant. In section 5, the decomposability or non-decomposability of the introduced EWs is discussed. In section 6 by using the approximate LP, two new kinds of mutispinor EWs are manipulated. Section 7 is devoted to a brief discussion about systems

with the odd number of the spinors. The paper is ended with a brief conclusion and five appendices.

2 Preliminaries

In this section, we briefly mention those concepts and subjects such as definitions and properties related to the EWs and their manipulation via the LP method as will be needed in the sequel; a more detailed treatment may be found, for example, in [48, 49].

2.1 Multipartite systems and Entanglement Witnesses

First we recall the notion of the separability for a system shared by N parties. Following Ref. [50], a k -partite split is a partition of the system into $k \leq N$ sets $\{S_i\}_{i=1}^k$, where each of them may be composed of several original parties. Given a density operator $\rho_{1\dots k} \in \mathcal{B}(H_1 \otimes \dots \otimes H_k)$ the Hilbert space of bounded operators acting on $H_1 \otimes \dots \otimes H_k$ associated with some k -partite split, we say that $\rho_{1\dots k}$ is a m -separable state if it is possible to find a convex decomposition for it such that in each pure state term at most m parties are entangled among each other, but not with any member of the other group of $N - m$ parties. For example, every 1-separable state, also called fully separable, can be written as

$$\rho_{1\dots k} = \sum_i p_i |\psi_i\rangle_1 \langle \psi_i| \otimes \dots \otimes |\psi_i\rangle_k \langle \psi_i| \quad (2.1)$$

with $p_i \geq 0$ and $\sum_i p_i = 1$, hence, the set of all fully separable states (hereafter, the separable states mean the fully separable states) is a convex set called the convex set of separable states (*CSSS*).

Definition 1. A Hermitian operator W is called an EW detecting the entangled state ρ_e if $\text{Tr}(W\rho_e) < 0$ and $\text{Tr}(W\rho_s) \geq 0$ for all separable states ρ_s .

This definition has a clear geometrical meaning. The expectation value of an observable depends linearly on the state. Thus, the set of states for which $\text{Tr}(W\rho) = 0$ holds is a

hyperplane in the set of all states, cutting this set into two parts. The part with $\text{Tr}(W\rho) > 0$ contains the set of all separable states where the other part (with $\text{Tr}(W\rho) < 0$) is the set of states detectable by W . From this geometrical interpretation it follows that for each entangled state ρ_e , there exists an EW detecting it [51].

Definition 2. An EW W is decomposable (d-EW) iff there exists operators P, Q_i with

$$W = P + Q_1^{T_A} + Q_2^{T_B} + \dots + Q_N^{T_Z} \quad P, Q_i \geq 0 \quad (2.2)$$

where superscripts T_i denote partial transposition with respect to the subsystem i . W is non-decomposable EW if it can not be put in the form (2.2) (for more details see [52]).

One should notice that, only non-decomposable EWs can detect PPT entangled states [48]. Then, an EW is nondecomposable (nd-EW) iff there exists at least one PPT entangled state which the witness detects [48].

Definition 3. An EW W is said to be optimal and denoted by $W_{opt.}$ if for all positive operators P and $\varepsilon > 0$, the following new Hermitian operator

$$W_{new} = (1 + \varepsilon)W_{opt.} - \varepsilon P \quad (2.3)$$

is not anymore an EW [5].

Suppose that there is a positive operator P and $\epsilon \geq 0$ such that $W_{new} = W_{opt.} - \epsilon P$ is yet an EW. This means that if $\text{Tr}(W_{opt.}\rho_s) = 0$, then $\text{Tr}(P\rho_s) = 0$, for all separable states ρ_s . By using the fact that every separable state is convex combination of pure product states, one can take ρ_s as a pure product state $|\psi\rangle\langle\psi|$. Also, one can assume that the positive operator P is a pure projection operator, since an arbitrary positive operator can be written as convex combination of pure projection operators with positive coefficients.

2.2 Manipulating EWs by exact and approximate LP method

Consider a Hermitian operator W with some negative eigenvalues as

$$W = a_0 I + \sum_{i=1}^n a_i Q_i \quad (2.4)$$

where Q_i are Hermitian operators which will be considered as multiplications of the Dirac γ matrices, with $-1 \leq \text{Tr}(Q_i \rho_s) \leq 1$ for each separable state ρ_s and a_i 's are real parameters with $a_0 \geq 0$.

As ρ_s varies over $CSSS$, the map $P_i = \text{Tr}(Q_i \rho_s)$ maps $CSSS$ into a convex region called feasible region (inside the hypercube defined by $-1 \leq P_i \leq 1$). Now, we try to choose the real parameters a_i , $i = 1, \dots, n$ (the allowed values of a_i define a region called EW's region in the space of the parameters a_i) such that the operator W given in (2.4) possesses at least one negative eigenvalue and its expectation value over any separable state be non-negative, i.e., the condition $\text{Tr}(W \rho_s) = a_0 + \sum_{i=1}^n a_i P_i \geq 0$ be satisfied for all P_i belonging to the feasible region. The region of the parameter space where, W possesses non-negative expectation value over all separable states (containing the EWs' region), is called the region of separable states non-negative expectation valued (denoted by SSNNEV).

Therefore, for determination of EWs of type (2.4), one needs to determine the minimum value of $a_0 + \sum_{i=1}^n a_i P_i$ over the feasible region (the minimum value must be non-negative) and hence the problem reduces to the optimization of the linear function $a_0 + \sum_{i=1}^n a_i P_i$ over the convex set of feasible region.

We note that, the minimum value of $F_W := \text{Tr}(W \rho_s)$ achieves for pure product states, since every separable state ρ_s can be written as a convex combination of pure product states (due to the convexity of separable region) as $\rho_s = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $p_i \geq 0$ and $\sum_i p_i = 1$, hence we have

$$F_W = \sum_i p_i \text{Tr}(W |\psi_i\rangle\langle\psi_i|) \geq C_{min} \quad (2.5)$$

with $C_{min} = \min_{|\psi\rangle \in D_{prod.}} \text{Tr}(W |\psi\rangle\langle\psi|)$, where $D_{prod.}$ denotes the set of pure product states.

Thus we need to find the pure product state $|\psi_{min}\rangle$ which minimizes $\text{Tr}(W|\psi\rangle\langle\psi|)$. For the cases that the feasible regions are simplexes (or at most convex polygons), the manipulation of the EWs amounts to

$$\begin{aligned} &\text{minimize} && F_W = a_0 + \sum_{i=1}^n a_i P_i \\ &\text{subject to} && \sum_{i=1}^n (c_{ij} P_i - d_i) \geq 0 \quad j = 1, 2, \dots \end{aligned} \quad (2.6)$$

where c_{ij} and d_i , $i, j = 1, 2, \dots$ are parameters of hyperplanes surrounding the feasible regions.

One can calculate the distributions P_i , consistent with the aforementioned optimization problem, from the information about the boundary of feasible region. To achieve the feasible region we obtain the extreme points corresponding to the product distributions P_i for every given product state by applying the special conditions on the parameters a_i . In fact, F_W themselves are functions of the product distributions, and they are in turn functions of ψ . They are not real variables of ψ but the product states will be multiplicative. If this feasible region constructs a polygon by itself, the corresponding boundary points of the convex hull will minimize exactly F_W in Eq.(2.6). This problem is called exact LP and the simplex method is the easiest way of solving it [45].

If the feasible region is not a polygon, with the help of tangent planes in this region at points which are determined either analytically or numerically, one can define a new convex hull which is a polygon encircling the feasible region. The points on the boundary of the polygon can approximately determine the minimum value of F_W in (2.6). Thus the approximated value is obtained via LP.

3 Entanglement witnesses for relativistic and non-relativistic multispinor systems in space-time dimension d

In this section, first we introduce our general formalism for constructing multispinor EWs by using Dirac γ matrices. In general we consider m spinors in the space-time of dimension d and D^m dimensional Hilbert space $\mathcal{H} = \underbrace{\mathcal{H}_D \otimes \dots \otimes \mathcal{H}_D}_m$ with $D = 2^{\lfloor d/2 \rfloor}$.

Let γ_μ , $\mu = 1, \dots, d$, be d Dirac γ matrices satisfying the anticommuting relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} I. \quad (3.7)$$

It follows from relations (3.7) that the γ matrices γ_μ generate an algebra which, as a vector space, has a dimension 2^d (for a brief review on the Dirac γ matrices see appendix A). We consider hermitian matrices A_i , $i = 1, 2, \dots, 2^d$ as all possible multiplications of γ_μ , $\mu = 1, 2, \dots, d$ up to multiplicative factors $\pm 1, \pm i$. Then, we will have

$$A_i^2 = I_{2^{\lfloor d/2 \rfloor}}, \quad A_i = A_i^\dagger, \quad \text{tr}(A_i A_j) = 2^{\lfloor d/2 \rfloor} \delta_{ij} \quad \text{for all } i, j = 1, 2, \dots, 2^d. \quad (3.8)$$

Clearly, the operators A_i , $i = 1, 2, \dots, 2^d$ either commute or anti-commute with each other, hence for even number of spinors, the matrices $\underbrace{A_i \otimes \dots \otimes A_i}_m$, $i = 1, \dots, 2^d$ commute with each other and can be diagonalized simultaneously. Also note that, we have $(A_i \otimes \dots \otimes A_i)^2 = I_{2^{m\lfloor d/2 \rfloor}}$, therefore the eigenvalues of $A_i \otimes \dots \otimes A_i$, $i = 1, \dots, 2^d$ are ± 1 . In order to construct Bell-states diagonal multispinor EWs, we will consider a hermitian operator W as superposition of the operators $A_i \otimes \dots \otimes A_i$, $i = 1, \dots, 2^d$ such that the conditions of definition 1 are satisfied. It will be seen that for some suitable superpositions of the operators $A_i \otimes \dots \otimes A_i$, the manipulation of the EWs reduces to the linear programming which can be solved exactly by using the simplex method.

It should be noticed that the linear combination of product of locally commuting matrices A_i , i.e.,

$$W = \sum_i a_i A_i^{(1)} \otimes \dots \otimes A_i^{(m)} \quad (3.9)$$

with $[A_i^{(k)}, A_j^{(k)}] = 0$ for $i \neq j$, $k = 1, \dots, m$ (the upper index (k) denotes the k -th spinor), can not be an EW, since its eigenvalues are all positive. In fact, W in (3.9) can be written as

$$W = \sum_{i; \alpha_1, \dots, \alpha_m} a_i \lambda_{i\alpha_1} \dots \lambda_{i\alpha_m} E_{\alpha_1}^{(1)} \otimes \dots \otimes E_{\alpha_m}^{(m)}, \quad (3.10)$$

where, $E_{\alpha_i}^{(k)}$, $k = 1, \dots, m$ is the projection operator to the eigenspace of $A_i^{(k)}$ corresponding to the eigenvalue $\lambda_{i\alpha_k}$. Then, the non-negativity of $\text{Tr}(W|\alpha_1\rangle\langle\alpha_1| \otimes \dots \otimes |\alpha_m\rangle\langle\alpha_m|) \geq 0$ implies that $\sum_i a_i \lambda_{i\alpha_1} \dots \lambda_{i\alpha_m} \geq 0$, i.e., the eigenvalues of W are all positive and so W can not be an EW.

3.1 Two particular sets of operators

In the following we choose two particular sets of the above introduced operators A_i , $i = 1, 2, \dots, 2^d$ which will be used in manipulating the multispinor EWs via exact LP optimization method.

3.1.1 First kind: Maximally anticommuting sets

As it was mentioned before, in the Hilbert space of dimension $D = 2^{\lfloor d/2 \rfloor}$, we have d matrices γ_μ , $\mu = 1, \dots, d$ which anticommute with each other. In the case of even dimension d , we denote $\gamma_S := i^{-d/2} \gamma_1 \gamma_2 \dots \gamma_d$ by γ_{d+1} , then the matrices $A_i = \gamma_i$, for $i = 1, 2, \dots, d, d+1$ form a maximally anticommuting set in the algebra of γ matrices (in the case of odd d , the set of matrices γ_i , $i = 1, \dots, d$ is maximally anticommuting set).

It is well known that every solution for the anticommutation relations (3.7) is equivalent to one another. That is if γ_μ and γ'_μ be two solutions for (3.7), then there exists a unitary matrix U such that

$$\gamma'_\mu = U \gamma_\mu U^{-1}, \quad (\mu = 1, 2, \dots, d). \quad (3.11)$$

For proof see Ref. [53]. Therefore, every EW defined as a superposition of the matrices γ_μ , can be replaced with another equivalent one in which the matrices γ_μ are replaced with the

matrices γ'_μ . Also, we will use the fact that, for any two anticommuting hermitian operators A and B , the expectation value of B over any eigenvector $|\psi\rangle$ of A with eigenvalue λ is zero and vice versa. Explicitly, we have

$$0 = \langle\psi|(AB + BA)|\psi\rangle = 2\lambda\langle\psi|B|\psi\rangle. \quad (3.12)$$

3.1.2 Second kind: Commuting sets which anticommute with each other

The second kind of sets for which the EW can be manipulated via exact LP, is the sets which are the union of three commuting sets C_1, C_2, C_3 such that $\{C_i, C_j\} = 0$ for $i \neq j$, i.e., for each $x, y \in C_i$, $i = 1, 2, 3$ we have $[x, y] = 0$, while for each $x \in C_i$, $z \in C_j$, $j \neq i$ we have $\{x, z\} = 0$.

Clearly, for a given d , there are several such commuting sets. In this paper, we will consider the following commuting sets C_i , $i = 1, 2, 3$ for constructing the multispinor EWs of the second kind:

$$\begin{aligned} C_1 &= \{-i\gamma_1\gamma_2, -\gamma_1\gamma_2\gamma_3\gamma_4, \dots, i^{-\lfloor d/2 \rfloor} \gamma_1\gamma_2 \dots \gamma_{2\lfloor d/2 \rfloor}\}, \quad C_2 = \{\gamma_1, i\gamma_1\gamma_3\gamma_4, \dots, i^{-\lfloor d/2 \rfloor - 1} \gamma_1\gamma_3 \dots \gamma_{2\lfloor d/2 \rfloor}\}, \\ C_3 &= \{\gamma_2, -i\gamma_2\gamma_3\gamma_4, \dots, i^{-\lfloor d/2 \rfloor + 1} \gamma_2\gamma_3 \dots \gamma_{2\lfloor d/2 \rfloor}\}. \end{aligned} \quad (3.13)$$

Note that each set C_i , $i = 1, 2, 3$ has cardinality $\lfloor d/2 \rfloor$.

3.2 Construction of BSD multispinor EWs

In this section we consider m d -dimensional spinors in the Hilbert space of dimension $2^{m\lfloor d/2 \rfloor}$ and construct EWs by using the two sets of hermitian operators introduced in the previous subsection. In the following, we will consider only the case of even m in details, where all of the discussions can be applied in the case of odd m , straightforwardly. In section 7, we will discuss the case of odd m briefly. Also, in the rest of the paper, we will consider even space-time dimensions d in order to simplify the notations. All of discussions and the equations given for even d such as the form of the introduced EWs, density matrices, etc. are the same for odd dimensions only by replacing d with $d - 1$.

3.2.1 EWs of the first kind

In the case of even m , we will consider the following hermitian matrix

$$W^{(m;d)} = a_0 I_{2^{md/2}} + \sum_{i=1}^{d+1} a_i \underbrace{\gamma_i^{(d)} \otimes \gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}}_m, \quad (3.14)$$

where, $\gamma_i^{(d)}$ for $i = 1, \dots, d+1$ are Dirac γ matrices in the space-time of even dimension d . In order that the observable (3.14) turns to an EW, we need to choose its parameters in such a way that it becomes a non-positive operator with non-negative expectation values over any separable state ρ_s .

Now it is the time to reduce the problem to the LP one. In order to determine the feasible region, we need to know the apexes, namely the extreme points, to construct the hyperplanes surrounding the feasible region.

For a given separable state ρ_s , the non-negativity of

$$\text{Tr}(W^{(m;d)} \rho_s) \geq 0, \quad (3.15)$$

implies that

$$a_0 + \sum_{i=1}^{d+1} a_i P_i \geq 0, \quad (3.16)$$

with

$$P_i = \text{tr}(\rho_s \gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}), \quad (3.17)$$

where all of the P_i 's lie in the interval $[-1, 1]$ (since, the eigenvalues of $\gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}$ are ± 1). Now, by using the fact that $\gamma_i^{(d)}$'s anticommute with each other and therefore the expectation value of $\gamma_j^{(d)}$ over any eigenvector of $\gamma_i^{(d)}$, $i \neq j$ is zero, one can deduce that the extremum

points or apexes are given as follows

Product state	$(P_1, P_2, \dots, P_{d+1})$
$ \psi_{\pm}^{(1)}\rangle$	$(\pm 1, 0, 0, 0, \dots, 0, 0, 0)$
$ \psi_{\pm}^{(2)}\rangle$	$(0, \pm 1, 0, 0, \dots, 0, 0, 0)$
\vdots	\vdots
$ \psi_{\pm}^{(k)}\rangle$	$(0, \dots, 0, \underbrace{\pm 1}_{k\text{-th}}, 0, \dots, 0)$
\vdots	\vdots
$ \psi_{\pm}^{(d+1)}\rangle$	$(0, 0, 0, \dots, 0, 0, 0, \pm 1)$

(3.18)

where, $|\psi_{\pm}^{(i)}\rangle$ are eigenvectors of $\gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}$ with eigenvalues ± 1 .

Regarding the above consideration, we are now ready to state the feasible region which is the convex hull of the apexes given in (3.18). According to the following inequalities

$$\text{Tr}\{\rho_s(I + \sum_{k=1}^{d+1} (-1)^{i_k} \underbrace{\gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}}_m)\} = 1 + \sum_{k=1}^{d+1} (-1)^{i_k} P_k \geq 0, \quad \forall (i_1, i_2, \dots, i_d, i_{d+1}) \in \{0, 1\}^{d+1},$$
(3.19)

(for the proof, see appendix B) any separable state is mapped into halfspaces defined by $1 + \sum_{k=1}^{d+1} (-1)^{i_k} P_k \geq 0$ and consequently, the feasible region corresponds to the intersection of these halfspaces which is the convex hull of the apexes. Therefore, the feasible region is surrounded by 2^{d+1} hyperplanes defined in a space of dimension $d+1$ as follows

$$1 + \sum_{k=1}^{d+1} (-1)^{i_k} P_k = 0 \quad , \quad \forall (i_1, i_2, \dots, i_d, i_{d+1}) \in \{0, 1\}^{d+1}.$$
(3.20)

Now, according to the prescription of subsection 2.2, namely the equation (2.6), the non-negativity of $W^{(m;d)}$ over separable states can be achieved by solving the following LP problem

$$\begin{aligned} & \text{minimize} && a_0 + \sum_{i=1}^{d+1} a_i P_i \\ & \text{subject to} && \begin{cases} 1 + \sum_{k=1}^{d+1} (-1)^{i_k} P_k \geq 0 \\ \forall |P_k| \leq 1, \quad k = 1, \dots, d, d+1 \end{cases} \end{aligned}$$
(3.21)

with $i_k \in \{0, 1\}$.

In the appendix C it is shown that any vertex point of the feasible region corresponds to a hyperplane of the region of SSNNEV and each hyperplane corresponding to the feasible region (e.g., each of the 2^{d+1} hyperplanes given in (3.20)) corresponds to an extreme point of this region. Therefore, by substitution of vertex points of the feasible region given in (3.18), we get the region of SSNNEV as the intersection of the following halfspaces

$$|a_i| \leq a_0, \quad i = 1, 2, \dots, d+1. \quad (3.22)$$

The above inequalities imply that in the space of parameters a_i of EWs, by fixing a_0 , all of the other a_i 's lie inside the hypercube $|a_i| \leq a_0$, $i = 1, \dots, d+1$. Also, we will need all eigenvalues of $W^{(m;d)}$ which consist of

$$\lambda_{i_1 \dots i_d}^{(m;d)} = a_0 + \sum_{k=1}^d (-1)^{i_k} a_k + i^{-md/2} (-1)^{i_1+i_2+\dots+i_d} a_{d+1}, \quad (3.23)$$

where $i_k \in \{0, 1\}$, $k = 1, 2, \dots, d$. Therefore, at least one of the eigenvalues $\lambda_{i_1 \dots i_d}^{(m;d)}$ must be negative to be guaranteed $W^{(m;d)}$ is an EW. We note that, the intersection of 2^d halfspaces defined by $\lambda_{i_1 \dots i_d}^{(m;d)} \geq 0$ is the region of $W^{(m;d)} \geq 0$ which is a polytope. Then, the complement of this polytope in the $d+1$ dimensional hypercube defined by $|a_i| \leq a_0$, $i = 1, 2, \dots, d+1$ is the region of EWs (clearly, the region of EWs is nonempty since $2^d < 2^{d+1}$). Also, it can be noticed that the optimal EWs are the farthest ones from the region $W^{(m;d)} \geq 0$, i.e., the vertex points of the the EWs' region.

Moreover, Eq.(3.19) shows that the region of SSNNEV (hypercube) has 2^{d+1} extreme points as $((-1)^{i_1}, \dots, (-1)^{i_d}, (-1)^{i_{d+1}})$ with $i_1, \dots, i_{d+1} \in \{0, 1\}$. In fact, the half of these points corresponds to the positive operators, where the other half of them corresponds to the extreme points of the EW's region, i.e., optimal EWs. These 2^d extreme points are given by $((-1)^{i_1}, (-1)^{i_2}, \dots, (-1)^{i_d}, -(-i)^{md/2} (-1)^{i_1+\dots+i_d})$ with $i_1, \dots, i_d \in \{0, 1\}$ corresponding to the

following 2^d optimal EWs:

$$W_{opt.}^{(m;d;i_1,\dots,i_d)} = I_{2^{md/2}} + \sum_{k=1}^d (-1)^{i_k} \underbrace{\gamma_k^{(d)} \otimes \dots \otimes \gamma_k^{(d)}}_m - (-i)^{md/2} (-1)^{i_1+\dots+i_d} \underbrace{\gamma_{d+1}^{(d)} \otimes \dots \otimes \gamma_{d+1}^{(d)}}_m, \quad (3.24)$$

where, $i_1, \dots, i_d \in \{0, 1\}$. We will prove the optimality of $W_{opt.}^{(m;d;i_1,\dots,i_d)}$ in subsection 3.3.

3.2.2 EWs of the second kind

Now, we consider a superposition of the second set of operators introduced in subsection 3.1.2 as follows

$$W'^{(m;d)} = a'_0 I_{2^{md/2}} + \sum_{i=1}^{3d/2} a'_i \underbrace{A'_i \otimes A'_i \otimes \dots \otimes A'_i}_m, \quad (3.25)$$

where, $A'_1, \dots, A'_{d/2} \in C_1$, $A'_{d/2+1}, \dots, A'_d \in C_2$ and $A'_{d+1}, \dots, A'_{3d/2} \in C_3$. Note that these $3d/2$ operators do not form independent generating set, namely, we have

$$A'_{d/2+i} = (-1)^{i-1} A'_{d/2+1} A'_i A'_1, \quad A'_{d+i} = i A'_{d/2+1} A'_i \quad \text{for } i = 1, 2, \dots, d/2. \quad (3.26)$$

In order that $W'^{(m;d)}$ be an EW, the expectation value of it on any separable state must be non-negative, i.e., for a given separable state ρ_s , the condition

$$a'_0 + \sum_{i=1}^{3d/2} a'_i P'_i \geq 0, \quad (3.27)$$

must be hold where, $P'_i := \text{tr}(\rho_s A'_i \otimes \dots \otimes A'_i)$. Clearly we have $|P'_i| \leq 1$ for $i = 1, \dots, 3d/2$, since the eigenvalues of $A'_i \otimes \dots \otimes A'_i$ are ± 1 .

The extremum points or apexes are given by

Product state	$(P'_1, \dots, P'_{d/2}; P'_{d/2+1}, \dots, P'_d; P'_{d+1}, \dots, P'_{3d/2})$
$ \psi_{\pm}^{(1;1)}\rangle$	$(\pm 1, 1, 1, \dots, 1; 0, 0, \dots, 0; 0, 0, \dots, 0)$
\vdots	\vdots
$ \psi_{\pm}^{(1;d/2)}\rangle$	$(1, \dots, 1, 1, \pm 1; 0, 0, \dots, 0; 0, \dots, 0, 0)$
$ \psi_{\pm}^{(2;d/2+1)}\rangle$	$(0, 0, \dots, 0; \pm 1, 1, 1, \dots, 1; 0, 0, \dots, 0)$
\vdots	\vdots
$ \psi_{\pm}^{(2;d)}\rangle$	$(0, 0, \dots, 0; 1, 1, \dots, 1, \pm 1; 0, 0, \dots, 0)$
$ \psi_{\pm}^{(3;d+1)}\rangle$	$(0, 0, \dots, 0; 0, \dots, 0, 0; \pm 1, 1, \dots, 1, 1)$
\vdots	\vdots
$ \psi_{\pm}^{(3;3d/2)}\rangle$	$(0, 0, \dots, 0; 0, 0, \dots, 0; 1, \dots, 1, 1, \pm 1)$

(3.28)

where, $|\psi_{\pm}^{(i;k)}\rangle$ for $i = 1, 2, 3$; $k = 1 + (i-1)d/2, \dots, id/2$ are common eigenvectors of the elements of the commuting set C_i such that

$$A'_j |\psi_{\pm}^{(i;k)}\rangle = (\pm 1)^{\delta_{jk}} |\psi_{\pm}^{(i;k)}\rangle, \quad A'_j \in C_i. \quad (3.29)$$

Note that, we have used the fact that the elements of C_i commute with each other and anti-commute with the elements of C_j , for $j \neq i$.

Considering the apexes given by (3.28), one can obtain the following inequalities

$$\begin{aligned} \text{Tr}\{\rho_s(I + (-1)^{i_1} A'_j \otimes \dots \otimes A'_j + (-1)^{i_2} A'_{j+d/2} \otimes \dots \otimes A'_{j+d/2} + (-1)^{i_3} A'_{j+d} \otimes \dots \otimes A'_{j+d})\} = \\ 1 + (-1)^{i_1} P'_j + (-1)^{i_2} P'_{j+d/2} + (-1)^{i_3} P'_{j+d} \geq 0, \end{aligned} \quad (3.30)$$

where, $i_1, i_2, i_3 \in \{0, 1\}$ and $j \in \{1, \dots, d/2\}$ (for the proof, see appendix B). Therefore, the feasible region is the intersection of the halfspaces given by (3.30) and the hyperplanes surrounding the feasible region are as follows

$$1 + (-1)^{i_1} P'_j + (-1)^{i_2} P'_{j+d/2} + (-1)^{i_3} P'_{j+d} = 0. \quad (3.31)$$

Again, in order to manipulate the EWs, according to the equation (2.6) one needs to solve the following LP problem

$$\begin{aligned} & \text{minimize} && a'_0 + \sum_{i=1}^{3d/2} a'_i P'_i \\ & \text{subject to} && \begin{cases} 1 + (-1)^{i_1} P'_j + (-1)^{i_2} P'_{j+d/2} + (-1)^{i_3} P'_{j+d} \geq 0 \\ \forall |P'_k| \leq 1, \end{cases} \end{aligned} \quad (3.32)$$

with $i_1, i_2, i_3 \in \{0, 1\}$ and $j \in \{1, \dots, d/2\}$.

Putting the coordinates of the apexes of the feasible region given by (3.28) in Eq.(3.27), yields the region of SSNNEV as the intersection of the following halfspaces

$$\left| \sum_{k=1}^{d/2} (-1)^{i_k} a'_k \right| \leq a'_0, \quad \left| \sum_{k=1}^{d/2} (-1)^{i_k} a'_{d/2+k} \right| \leq a'_0, \quad \left| \sum_{k=1}^{d/2} (-1)^{i_k} a'_{d+k} \right| \leq a'_0. \quad (3.33)$$

We will also need all of the eigenvalues of $W'^{(m;d)}$ which consist of

$$\lambda'_{i_1 \dots i_{d/2+1}}(m;d) = a'_0 + \sum_{k=1}^{d/2+1} (-1)^{i_k} a'_k + \sum_{k=2}^{d/2} (-1)^{i_1+i_{d/2+1}+i_k} a'_{d/2+k} + \sum_{k=1}^{d/2} (-1)^{m/2+i_{d/2+1}+i_k} a'_{d+k}, \quad (3.34)$$

where $i_1, \dots, i_{d/2+1} \in \{0, 1\}$ (we have used the Eq.(3.26)). Again, in order that $W'^{(m;d)}$ be an EW, at least one of the eigenvalues $\lambda'_{i_1 \dots i_{d/2+1}}(m;d)$ must be negative. In fact, the intersection of $2^{d/2+1}$ halfspaces defined by $\lambda'_{i_1 \dots i_{d/2+1}}(m;d) \geq 0$ is the region of $W'^{(m;d)} \geq 0$ which is a polytope. Then, the complement of this polytope in the region defined by (3.33) is the region of EWs.

Also, the inequalities (3.30) imply that the region of SSNNEV has $8 \cdot d/2 = 4d$ vertices as $(0, \dots, 0, \underbrace{(-1)^{i_1}}_j, 0, \dots, 0; 0, \dots, 0, \underbrace{(-1)^{i_2}}_{j+d/2}, 0, \dots, 0; 0, \dots, 0, \underbrace{(-1)^{i_3}}_{j+d}, 0, \dots, 0)$ with $j \in \{1, \dots, d/2\}$ and $i_1, i_2, i_3 \in \{0, 1\}$. It can be shown that, half of these points corresponds to the positive operators, where the other half corresponds to the optimal EWs. In fact we have $2d$ extreme points as $(0, \dots, 0, \underbrace{(-1)^{i_1}}_j, 0, \dots, 0; 0, \dots, 0, \underbrace{(-1)^{i_2}}_{j+d/2}, 0, \dots, 0; 0, \dots, 0, \underbrace{-(-1)^{m/2+i_1+i_2}}_{j+d}, 0, \dots, 0)$ with $j \in \{1, \dots, d/2\}$ and $i_1, i_2 \in \{0, 1\}$ corresponding to the following $2d$ optimal EWs:

$$W_{opt.}^{(m;d;i_1,i_2;j)} = I + (-1)^{i_1} \underbrace{A'_j \otimes \dots \otimes A'_j}_m + (-1)^{i_2} \underbrace{A'_{j+d/2} \otimes \dots \otimes A'_{j+d/2}}_m - (-1)^{m/2+i_1+i_2} \underbrace{A'_{j+d} \otimes \dots \otimes A'_{j+d}}_m, \quad (3.35)$$

with $i_1, i_2 \in \{0, 1\}$, $j \in \{1, \dots, d/2\}$. We prove the optimality of these EWs in the following subsection.

3.3 Optimality of EWs $W_{opt.}^{(m;d;i_1,\dots,i_d)}$ and $W_{opt.}'^{(m;d;i_1,i_2;j)}$

In this section, we prove the optimality of EWs $W_{opt.}^{(m;d;i_1,\dots,i_d)}$ and $W_{opt.}'^{(m;d;i_1,i_2;j)}$ given by (3.24) and (3.35), respectively.

3.3.1 Optimality of $W_{opt.}^{(m;d;i_1,\dots,i_d)}$

In order to prove that the $W_{opt.}^{(m;d;i_1,\dots,i_d)}$ given in Eq.(3.24) is optimal, we first rewrite $W_{opt.}^{(m;d;i_1,\dots,i_d)}$ as follows

$$W_{opt.}^{(m;d;i_1,\dots,i_d)} = I + \sum_{k=1}^d (-1)^{i_k} O_k - (-i)^{md/2} (-1)^{i_1+\dots+i_d} O_{d+1}, \quad (3.36)$$

where, $O_i := \underbrace{\gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}}_m$ for $i = 1, 2, \dots, d, d+1$ and prove the optimality of

$$W_{opt.}^{(m;d;1,\dots,1)} = I - \sum_{k=1}^d O_k - (-i)^{md/2} O_{d+1}, \quad (3.37)$$

where the optimality of the other cases can be proved similarly. According to the definition 3 of subsection 2.1, it suffices to show that there exists no positive operator P such that $W_{new} := (1 + \varepsilon)W_{opt.}^{(m;d;1,\dots,1)} - \varepsilon P$ be an EW, namely it must be proved that for any pure product state $|\psi\rangle$ such that $Tr(W_{opt.}^{(m;d;1,\dots,1)}|\psi\rangle\langle\psi|) = 0$, there exists no positive operator P with the constraint $Tr(P|\psi\rangle\langle\psi|) = 0$. To this end, first we note that the expectation value of the operator $W_{opt.}^{(m;d;1,\dots,1)}$ in (3.37) over pure product states $|\psi\rangle$ will vanish if one of the equations

$$O_i|\psi\rangle = |\psi\rangle \text{ for some } i = 1, 2, \dots, d \text{ or}$$

$$O_{d+1}|\psi\rangle = (-i)^{md/2}|\psi\rangle \quad (3.38)$$

be satisfied (recall that $\langle\psi|O_j|\psi\rangle = 0$, for $j \neq i$, since $|\psi\rangle$ is a product state). Regarding the definition 3 of subsection 2.1, we may assume that the positive operator P is a pure projection

operator, since any arbitrary positive operator can be written as convex combination of pure projection operators with positive coefficients. The equations (3.37) and (3.38) indicate that, in order that $Tr(P|\psi\rangle\langle\psi|) = 0$ be satisfied, the operator P must be the projection operator to the eigenspace of O_i , $i = 1, 2, \dots, d$ with eigenvalue -1 and O_{d+1} with eigenvalue $-(-i)^{md/2}$. But from the fact that $O_{d+1} = (-i)^{md/2}O_1\dots O_d$, if $|\psi'\rangle$ be the common eigenket of the operators O_1, O_2, \dots, O_d with eigenvalue -1 , then $|\psi'\rangle$ will be an eigenket of O_{d+1} with eigenvalue $(-i)^{md/2}$ ($O_{d+1}|\psi'\rangle = (-i)^{md/2}(-1)^d|\psi'\rangle = (-i)^{md/2}|\psi'\rangle$). Therefore, the eigenspace of O_i , $i = 1, 2, \dots, d$ with eigenvalue -1 and O_{d+1} with eigenvalue $-(-i)^{md/2}$ is a null space.

3.3.2 Optimality of $W_{opt.}^{(m;d;i_1,i_2;j)}$

We prove the optimality of $W_{opt.}^{(m;d;i_1,i_2;j)}$ for $j = 1$ and $i_1 = i_2 = 1$, the optimality of the other cases can be proved similarly. For $j = 1$ and $i_1 = i_2 = 1$, we have

$$W_{opt.}^{(m;d;1,1;1)} = I - A'_1 \otimes \dots \otimes A'_1 - A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1} - (-1)^{m/2} A'_{d+1} \otimes \dots \otimes A'_{d+1}. \quad (3.39)$$

As regards the arguments of subsection 3.3.1, we need to show that the eigenspace of $A'_1 \otimes \dots \otimes A'_1$, $A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1}$ with eigenvalue -1 and $A'_{d+1} \otimes \dots \otimes A'_{d+1}$ with eigenvalue $-(-1)^{m/2}$ is a null space. Assume that $|\psi'\rangle$ be the eigenket of $A'_1 \otimes \dots \otimes A'_1$ and $A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1}$ with eigenvalue -1 , then by using (3.26) we have

$$A'_{d+1} \otimes \dots \otimes A'_{d+1}|\psi'\rangle = i^m A'_{d/2+1} A'_1 \otimes \dots \otimes A'_{d/2+1} A'_1 |\psi'\rangle = (-1)^{m/2} |\psi'\rangle. \quad (3.40)$$

This implies that, every eigenstate of $A'_1 \otimes \dots \otimes A'_1$ and $A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1}$ with eigenvalue -1 is necessarily an eigenstate of $A'_{d+1} \otimes \dots \otimes A'_{d+1}$ with eigenvalue $(-1)^{m/2}$ and so the corresponding common eigenspace is a null space.

4 Entangled states which can be detected by BSD multispinor EWs

In this section, we discuss the Bell-states diagonal entangled states which can be detected by the introduced EWs. To do so, first we consider the most significant case of bipartite system of spinors in four-dimensional space-time and then generalize the discussions to multipartite higher dimensional cases. In the bipartite case, we use the Weyl or chiral representation of the gamma matrices and follow the notation of the text by Weinberg [54] to take the Lorentz transformation of states more conveniently. In the case of EWs of the first kind with $m = 2, d = 4$, we consider both the relativistic and non-relativistic BSD density matrices in order to discuss the effect of the Lorentz transformation on the amount of entanglement measured by the Hilbert-Schmidt measure, where for the case of EWs of the second kind with $d = 4$ and EWs with $d > 4$, we discuss only the non-relativistic density matrices which can be detected by these EWs (and do not deal with the amount of entanglement), where the discussions about relativistic case in $d = 4$ can be generalized straightforwardly to the cases $d > 4$.

4.1 Entanglement properties of relativistic and non-relativistic BSD density matrices in four-dimensional space-time

In order to define some interesting entangled states detectable by the introduced EWs, we construct Bell-type and iso-concurrence type entangled states and investigate their entanglement properties by using the introduced EWs (entanglement properties of non-relativistic Bell-diagonal states and iso-concurrence states have been studied in [8]- [11]). To this end, we will take the helicity basis (simultaneously eigenstates of the helicity operator [17] and $\gamma'_5 = (H \otimes I)\gamma_5(H \otimes I)$, with $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$ known as Hadamard transform) and construct Bell-type and iso-concurrence type entangled states by considering their combinations.

It is well known that, the helicity eigenstates [17] are given by

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad |\psi_4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (4.41)$$

the first two of which correspond to positive energy, and the second two to negative energy. One could notice that, the helicity eigenstates $|\psi_2\rangle, |\psi_3\rangle$ and $|\psi_4\rangle$ can be obtained from $|\psi_1\rangle$ by local unitary transformations as follows

$$|\psi_2\rangle = (\sigma_z \otimes \sigma_x)|\psi_1\rangle, \quad |\psi_3\rangle = (\sigma_z \otimes I)|\psi_1\rangle, \quad |\psi_4\rangle = (I \otimes \sigma_x)|\psi_1\rangle. \quad (4.42)$$

Now, we define the following Bell states:

$$\begin{aligned} |\psi_{\pm}\rangle^{(1,2)} &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_1\rangle \pm |\psi_2\rangle|\psi_2\rangle), & |\phi_{\pm}\rangle^{(1,2)} &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_2\rangle \pm |\psi_2\rangle|\psi_1\rangle), \\ |\psi_{\pm}\rangle^{(3,4)} &= \frac{1}{\sqrt{2}}(|\psi_3\rangle|\psi_3\rangle \pm |\psi_4\rangle|\psi_4\rangle), & |\phi_{\pm}\rangle^{(3,4)} &= \frac{1}{\sqrt{2}}(|\psi_3\rangle|\psi_4\rangle \pm |\psi_4\rangle|\psi_3\rangle), \\ |\psi_{\pm}\rangle^{(1,3)} &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_1\rangle \pm |\psi_3\rangle|\psi_3\rangle), & |\phi_{\pm}\rangle^{(1,3)} &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_3\rangle \pm |\psi_3\rangle|\psi_1\rangle), \\ |\psi_{\pm}\rangle^{(2,4)} &= \frac{1}{\sqrt{2}}(|\psi_2\rangle|\psi_2\rangle \pm |\psi_4\rangle|\psi_4\rangle), & |\phi_{\pm}\rangle^{(2,4)} &= \frac{1}{\sqrt{2}}(|\psi_2\rangle|\psi_4\rangle \pm |\psi_4\rangle|\psi_2\rangle), \\ |\psi_{\pm}\rangle^{(1,4)} &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_1\rangle \pm |\psi_4\rangle|\psi_4\rangle), & |\phi_{\pm}\rangle^{(1,4)} &= \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_4\rangle \pm |\psi_4\rangle|\psi_1\rangle), \\ |\psi_{\pm}\rangle^{(2,3)} &= \frac{1}{\sqrt{2}}(|\psi_2\rangle|\psi_2\rangle \pm |\psi_3\rangle|\psi_3\rangle), & |\phi_{\pm}\rangle^{(2,3)} &= \frac{1}{\sqrt{2}}(|\psi_2\rangle|\psi_3\rangle \pm |\psi_3\rangle|\psi_2\rangle) \end{aligned} \quad (4.43)$$

and introduce the following 16 orthonormal entangled states as follows:

$$\begin{aligned} |\Phi^1\rangle &= \cos\theta|\psi_+\rangle^{(1,2)} + \sin\theta|\psi_+\rangle^{(3,4)}, & |\Phi^2\rangle &= -\sin\theta|\psi_+\rangle^{(1,2)} + \cos\theta|\psi_+\rangle^{(3,4)}, \\ |\Phi^3\rangle &= \cos\theta|\psi_-\rangle^{(1,2)} + \sin\theta|\psi_-\rangle^{(3,4)}, & |\Phi^4\rangle &= -\sin\theta|\psi_-\rangle^{(1,2)} + \cos\theta|\psi_-\rangle^{(3,4)}, \\ |\Phi^5\rangle &= \cos\theta|\phi_+\rangle^{(1,2)} + \sin\theta|\phi_+\rangle^{(3,4)}, & |\Phi^6\rangle &= -\sin\theta|\phi_+\rangle^{(1,2)} + \cos\theta|\phi_+\rangle^{(3,4)}, \end{aligned}$$

$$\begin{aligned}
|\Phi^7\rangle &= \cos\theta|\phi_-\rangle^{(1,2)} + \sin\theta|\phi_-\rangle^{(3,4)}, & |\Phi^8\rangle &= -\sin\theta|\phi_-\rangle^{(1,2)} + \cos\theta|\phi_-\rangle^{(3,4)}, \\
|\Phi^9\rangle &= \cos\theta|\phi_+\rangle^{(1,3)} + \sin\theta|\phi_+\rangle^{(2,4)}, & |\Phi^{10}\rangle &= -\sin\theta|\phi_+\rangle^{(1,3)} + \cos\theta|\phi_+\rangle^{(2,4)}, \\
|\Phi^{11}\rangle &= \cos\theta|\phi_-\rangle^{(1,3)} + \sin\theta|\phi_-\rangle^{(2,4)}, & |\Phi^{12}\rangle &= -\sin\theta|\phi_-\rangle^{(1,3)} + \cos\theta|\phi_-\rangle^{(2,4)}, \\
|\Phi^{13}\rangle &= \cos\theta|\phi_+\rangle^{(1,4)} + \sin\theta|\phi_+\rangle^{(2,3)}, & |\Phi^{14}\rangle &= -\sin\theta|\phi_+\rangle^{(1,4)} + \cos\theta|\phi_+\rangle^{(2,3)}, \\
|\Phi^{15}\rangle &= \cos\theta|\phi_-\rangle^{(1,4)} + \sin\theta|\phi_-\rangle^{(2,3)}, & |\Phi^{16}\rangle &= -\sin\theta|\phi_-\rangle^{(1,4)} + \cos\theta|\phi_-\rangle^{(2,3)}. \quad (4.44)
\end{aligned}$$

Although we will not deal with the concurrence of these states, due to the similarity of these states to the iso-concurrence states in the two-qubit systems considered in [8]-[11], we refer to these states as iso-concurrence type states. We note that, for $\theta = \pi/4$ in (4.44) we obtain the so-called Bell-type states which are maximally entangled states. For example we have

$$Tr(W_{opt.}^{(2;4;i_1,\dots,i_4)}|\Phi^1\rangle\langle\Phi^1|) = 1 + \sin 2\theta[(-1)^{i_1} - (-1)^{i_2} + (-1)^{i_3} - (-1)^{i_4}] - (-1)^{i_1+\dots+i_4},$$

$$Tr(W_{opt.}'^{(2;4;i_1,i_2;1)}|\Phi^1\rangle\langle\Phi^1|) = 1 + (-1)^{i_1} + \sin 2\theta[(-1)^{i_2} - (-1)^{i_1+i_2}], \quad (4.45)$$

where, the most negative value of (4.45) is obtained for $\theta = \pi/4$ by taking $i_1 = i_3 = 1, i_2 = i_4 = 0$ in $W_{opt.}^{(2;4;i_1,\dots,i_4)}$ and $i_1 = i_2 = 1$ in $W_{opt.}'^{(2;4;i_1,i_2;1)}$, respectively.

We consider now the spinor “EPR state” [47] as follows

$$|\Psi(\vec{P}_1 = 0, \vec{P}_2 = 0)\rangle = \sqrt{\frac{m}{2}}(|\psi_4\rangle|\psi_1\rangle - i|\psi_1\rangle|\psi_4\rangle), \quad (4.46)$$

where, \vec{P} is three-momentum. This state corresponds to a Lorentz frame where both particles are at rest. As far as the detection of entanglement is concerned, the Lorentz transformation do not change the situation, since these transformations take product states to some another product ones [22] and so preserves the entanglement. We note that the “EPR state” (4.46) can be obtained from the state $|\phi_-\rangle^{(1,4)}$ by applying the rotation $S = e^{i\pi/4 I \otimes \sigma_z}$ on the first particle. It follows that, if an EW W can detect the state $|\phi_-\rangle^{(1,4)}$, then $(S \otimes I)W(S \otimes I)^{-1}$ will be detect the “EPR state” $|\Psi(\vec{P}_1 = 0, \vec{P}_2 = 0)\rangle$. Now, one can easily check that

$$Tr(W_{opt.}^{(2;4;i_1,\dots,i_4)}|\phi_-\rangle^{(1,4)}\langle\phi_-|^{(1,4)}) = 1 - (-1)^{i_1} - (-1)^{i_2} - (-1)^{i_3}, \quad (4.47)$$

which shows that $W_{opt.}^{(2;4;0,0,0,i_4)}$, $i_4 = 0, 1$ detect $|\phi_{-}\rangle^{(1,4)}$. By taking the similarity transformation $(S \otimes I)W_{opt.}^{(2;4;0,0,0,i_4)}(S \otimes I)^{-1}$, we obtain

$$\tilde{W}_{opt.}^{(2;4;0,0,0,i_4)} = (S \otimes I)W_{opt.}^{(2;4;0,0,0,i_4)}(S \otimes I)^{-1} = I - \gamma_2 \otimes \gamma_1 + \gamma_1 \otimes \gamma_2 + \gamma_3 \otimes \gamma_3 + (-1)^{i_4} \gamma_4 \otimes \gamma_4 - (-1)^{i_4} \gamma_5 \otimes \gamma_5, \quad (4.48)$$

where, we have used the equalities $S\gamma_1 S^{-1} = -\gamma_2$, $S\gamma_2 S^{-1} = \gamma_1$, $S\gamma_3 S^{-1} = \gamma_3$, $S\gamma_4 S^{-1} = \gamma_4$, $S\gamma_5 S^{-1} = \gamma_5$. Then, one can easily obtain $Tr(\tilde{W}_{opt.}^{(2;4;0,0,0,i_4)}|\Psi(\vec{P}_1 = 0, \vec{P}_2 = 0)\rangle\langle\Psi(\vec{P}_1 = 0, \vec{P}_2 = 0)|) = -2$.

Now, let $\rho_{BSD}(0)$ be a so called Bell-states diagonal (BSD) density matrix of a bipartite system in the rest frame which has the following decomposition

$$\rho_{BSD}(0) = \sum_{i=0}^{15} a_i |\Psi_i(0)\rangle\langle\Psi_i(0)|, \quad \sum_i a_i = 1, \quad a_i \geq 0 \quad (4.49)$$

where, $|\Psi_i(0)\rangle$, denote the Bell-type states obtained by taking $\theta = \pi/4$ in (4.44). In the appendix E, we show that any such BSD density matrix can be written as

$$\rho_{BSD}(0) = \frac{1}{16} I \otimes I + \sum_{\mu=0}^{14} b_\mu A_\mu \otimes A_\mu, \quad (4.50)$$

where, A_μ 's are given by

$$\begin{aligned} A_\mu &= \gamma^\mu; \mu = 0, 1, 2, 3 \quad A_4 = \gamma^5, \quad A_5 = \gamma^0 \gamma^1, \quad A_6 = \gamma^0 \gamma^2, \quad A_7 = -i\gamma^0 \gamma^3, \quad A_8 = i\gamma^1 \gamma^2, \\ A_9 &= -i\gamma^1 \gamma^3, \quad A_{10} = i\gamma^2 \gamma^3, \quad A_{11} = -i\gamma^0 \gamma^5, \quad A_{12} = \gamma^1 \gamma^5, \quad A_{13} = \gamma^2 \gamma^5, \quad A_{14} = \gamma^3 \gamma^5, \end{aligned} \quad (4.51)$$

such that γ^μ for $\mu = 0, 1, 2, 3, 5$ defined as

$$\gamma^0 = \sigma_x \otimes I, \quad \gamma^1 = i\sigma_y \otimes \sigma_x, \quad \gamma^2 = i\sigma_y \otimes \sigma_y, \quad \gamma^3 = i\sigma_y \otimes \sigma_z, \quad \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\sigma_z \otimes I, \quad (4.52)$$

are the gamma matrices in the chiral representation.

Clearly, the coefficients b_μ in (4.50) are given by $b_\mu = \frac{1}{16} Tr(\rho_{BSD}(0) A_\mu \otimes A_\mu)$. The positivity of $\rho_{BSD}(0)$ implies that

$$\lambda_{BD}^{i_1, \dots, i_4} = 1/16 + (-1)^{i_0} b_0 + (-1)^{i_1} b_1 + (-1)^{i_2} b_2 + (-1)^{i_3} b_3 - (-1)^{i_0 + \dots + i_3} b_4 + (-1)^{i_0 + i_1} b_5 + (-1)^{i_0 + i_2} b_6 -$$

$$\begin{aligned}
& (-1)^{i_0+i_3}b_7 - (-1)^{i_1+i_2}b_8 - (-1)^{i_1+i_3}b_9 - (-1)^{i_2+i_3}b_{10} + (-1)^{i_1+i_2+i_3}b_{11} - (-1)^{i_0+i_2+i_3}b_{12} - \\
& (-1)^{i_0+i_1+i_3}b_{13} - (-1)^{i_0+i_1+i_2}b_{14} \geq 0,
\end{aligned} \tag{4.53}$$

Moreover, by imposing the positivity of partial transposition of $\rho_{BSD}(0)$, we obtain 16 other inequalities as (4.53) in which the sign of the coefficients $b_1, b_3, b_6, b_9, b_{11}$ and b_{13} are opposite with those of (4.53). The region defined by intersection of these 32 halfspaces is a convex polytope which is the region of PPT density matrices, where its vertices can be obtained by maximizing the left hand side of one of the inequalities (corresponding to the halfspaces) subject to the other 31 inequalities as constraints (this can be done simply with the simplex method in maple). On the other hand, the intersection of halfspaces defined by

$$Tr(\rho_{BSD}(0)W_{opt.}^{(2;4;i_0,\dots,i_3)}) = 1 + 16[(-1)^{i_0}b_0 + (-1)^{i_1}b_1 + (-1)^{i_2}b_2 + (-1)^{i_3}b_3 - (-1)^{i_0+\dots+i_3}b_4] \geq 0, \tag{4.54}$$

form a convex polytope, where the intersection of its complement and the region of PPT density matrices, is the region of detectable PPT entangled states.

In order to simply determine the region of separable and PPT entangled states, we consider the special case of BSD density matrices which are written as

$$\rho_{BSD}(0) = \frac{1}{16}I_{16} + b_0\gamma^0 \otimes \gamma^0 + b_1\gamma^1 \otimes \gamma^1 + b_2\gamma^2 \otimes \gamma^2 + b_3\gamma^3 \otimes \gamma^3 + b_4\gamma^5 \otimes \gamma^5. \tag{4.55}$$

Then, the positivity condition (4.53) implies that

$$\frac{1}{16} + (-1)^{i_0}b_0 + (-1)^{i_1}b_1 + (-1)^{i_2}b_2 + (-1)^{i_3}b_3 - (-1)^{i_0+\dots+i_3}b_4 \geq 0. \tag{4.56}$$

The inequalities (4.56) define a polyhedron in the 5-dimensional space with coordinates $(b_0, b_1, b_2, b_3, b_4)$ where its vertices are as follows

$$\rho_{BSD}^{(i_0,\dots,i_3)}(0) = \frac{1}{48}((-1)^{i_0}, (-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}, -(-1)^{i_0+\dots+i_3}), \quad i_1, \dots, i_4 \in \{0, 1\}. \tag{4.57}$$

Now, one can easily show that the optimal EWs of the first kind in the case of $m = 2, d = 4$

are given by

$$W_{opt.}^{(2;4;i_0,\dots,i_3)} = I_{16} + (-1)^{i_0}\gamma^0 \otimes \gamma^0 + (-1)^{i_1}\gamma^1 \otimes \gamma^1 + (-1)^{i_2}\gamma^2 \otimes \gamma^2 + (-1)^{i_3}\gamma^3 \otimes \gamma^3 + (-1)^{i_0+\dots+i_3}\gamma^5 \otimes \gamma^5. \quad (4.58)$$

By using (4.57) and (4.58), one can obtain

$$Tr[W_{opt.}^{(2;4;i_0,\dots,i_3)} \rho_{BSD}^{(j_0,\dots,j_3)}(0)] = 1 + \frac{1}{3}[(-1)^{i_0+j_0} + (-1)^{i_1+j_1} + (-1)^{i_2+j_2} + (-1)^{i_3+j_3} - (-1)^{i_0+\dots+i_3+j_0+\dots+j_3}]. \quad (4.59)$$

Then, $\rho_{BSD}^{(j_0,\dots,j_3)}(0)$ can be detected by $W_{opt.}^{(2;4;1-j_0,\dots,1-j_3)}$, where

$$Tr[W_{opt.}^{(2;4;1-j_0,\dots,1-j_3)} \rho_{BSD}^{(j_0,\dots,j_3)}(0)] = 1 - \frac{5}{3} = -\frac{2}{3}. \quad (4.60)$$

In order to determine the region of entangled states, we consider the constraints defined by

$$Tr(\rho_{BSD}(0)W_{opt.}^{(2;4;i_0,\dots,i_3)}) = 16\left(\frac{1}{16} + (-1)^{i_0}b_0 + (-1)^{i_1}b_1 + (-1)^{i_2}b_2 + (-1)^{i_3}b_3 + (-1)^{i_0+\dots+i_3}b_4\right) \geq 0 \quad (4.61)$$

The inequalities (4.56) and (4.61) form a polyhedron with vertices $(\pm 1, 0, 0, 0, 0)$, $(0, \pm 1, 0, 0, 0)$ and $(0, 0, \pm 1, 0, 0)$, $(0, 0, 0, \pm 1, 0)$ and $(0, 0, 0, 0, \pm 1)$. It should be noticed that, these density matrices, i.e., $\frac{1}{16}(I \pm \gamma^i \otimes \gamma^i)$, $i = 0, 1, 2, 3, 5$ can be written as superposition of pure product states and hence are separable. Therefore, the polyhedron defined by 16 inequalities of (4.56) is divided to 17 regions: the central region which corresponds to the polyhedron defined by (4.56) and (4.61), is the region of separable states. The other 16 regions are in fact the smaller polyhedrons which are associated with the PPT entangled states. Each of these polyhedrons corresponds to an offence of one of the 16 inequalities in (4.61).

So far, we considered the BSD density matrices in the rest frame S , where the spinors are at rest. Now, we describe the situation where, the particles are moving with constant velocity with respect to each other. We take the spinor representation $D(L(p))$ of a standard boost $L(p)$ of rapidity ξ as

$$D(L(p)) = \exp\left\{-\frac{\xi}{2} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & -\vec{\sigma} \cdot \vec{p} \end{pmatrix}\right\} =$$

$$\cosh(\xi/2) \begin{pmatrix} 1 - p_3 \tanh(\xi/2) & -p_- \tanh(\xi/2) & 0 & 0 \\ -p_+ \tanh(\xi/2) & 1 + p_3 \tanh(\xi/2) & 0 & 0 \\ 0 & 0 & 1 + p_3 \tanh(\xi/2) & p_- \tanh(\xi/2) \\ 0 & 0 & p_+ \tanh(\xi/2) & 1 - p_3 \tanh(\xi/2) \end{pmatrix}. \quad (4.62)$$

In the above equation, $L(p)$ is a coordinate Lorentz transformation to a frame S' moving with velocity $v/c = \tanh(-\xi)$ such that from S' the particle at rest in frame S is observed to have velocity v/c . The vector $\vec{p} = (p_1, p_2, p_3)$ is a unit vector in the direction of \mathbf{p} with $p_{\pm} = p_1 \pm ip_2$.

We are now ready to describe the transformed spinors, by using the rest frame spinors in Eq.(4.41) as follows

$$|\psi_i(p)\rangle = \frac{1}{\sqrt{\cosh \xi}} D(L(p)) |\psi_i(0)\rangle, \quad (4.63)$$

($|\psi_i(0)\rangle \equiv |\psi_i\rangle$ are the helicity basis defined by Eq.(4.41)). Then, the rest frame BSD density matrices given by Eq.(4.49) are transformed as

$$\begin{aligned} \rho_{BSD}(\vec{p}) &= \frac{1}{\cosh^2(\xi)} [D(L(p)) \otimes D(L(p))] \rho_{BSD}(0) [D^\dagger(L(p)) \otimes D^\dagger(L(p))] = \\ &\sum_{i=0}^{15} a_i |\Psi_i(p)\rangle \langle \Psi_i(p)|, \quad \sum_i a_i = 1, \quad a_i \geq 0. \end{aligned} \quad (4.64)$$

In order to avoid more complexities, we consider the BSD density matrices given by (4.55) with $\vec{p} = (0, 0, 1)$. Then the Lorentz transformation $D(L(p))$ reads

$$D(L(p)) = D^\dagger(L(p)) = \cosh(\xi/2) (I \otimes I - \tanh(\xi/2) \sigma_z \otimes \sigma_z). \quad (4.65)$$

In the following, we discuss the effect of the Lorentz transformation (4.65) on the amount of entanglement. To do so, we will use the Hilbert-Schmidt measure of entanglement [40]. In order to define this measure, we recall that the Hilbert-Schmidt norm is defined as

$$\|A\| = \sqrt{\langle A, A \rangle}, \quad (4.66)$$

where, $\langle A, B \rangle = \text{Tr}(A^\dagger B)$. With help of the norm (4.66), the Hilbert-Schmidt distance between two arbitrary states ρ_1, ρ_2 can be defined as

$$d_{HS}(\rho_1, \rho_2) = \|\rho_1 - \rho_2\|. \quad (4.67)$$

By using the Hilbert-Schmidt distance, the so-called Hilbert-Schmidt measure of entanglement is defined as

$$D(\rho_{ent.}) = \min_{\rho \in S} \|\rho - \rho_{ent.}\|, \quad (4.68)$$

where, S is the set of separable states. In fact, the Hilbert-Schmidt measure is the minimal distance of an entangled state $\rho_{ent.}$ to the set of separable states.

For an entangled state $\rho_{ent.}$, the minimum of the Hilbert-Schmidt distance (the Hilbert-Schmidt measure) is attained for some state ρ_s since the norm is continuous and the set S is compact. Due to the Bertlmann-Narnhofer-Thirring Theorem [41], there exist an equivalence between the Hilbert-Schmidt measure and the concept of optimal entanglement witnesses as follows: The Hilbert-Schmidt measure of an entangled state equals the maximal violation of the inequality $\text{Tr}(W\rho) \geq 0$,

$$D(\rho_{ent.}) = \|\rho_s - \rho_{ent.}\| = -\langle \rho_{ent.}, W_{opt} \rangle = -\text{Tr}(\rho_{ent.} W_{opt}), \quad (4.69)$$

where,

$$W_{opt} = \frac{\rho_s - \rho_{ent.} - \langle \rho_s, \rho_s - \rho_{ent.} \rangle \mathbf{1}}{\|\rho_s - \rho_{ent.}\|} \quad (4.70)$$

is an optimal entanglement witness (for more details see Refs. [40], [41]). Therefore, in order to calculate the Hilbert-Schmidt measure for the PPT BSD entangled states in the rest frame given by Eq.(4.57), we will use the optimal EWs (4.58) and Eq.(4.70) to obtain the state ρ_s in (4.70). Then, by using the Lorentz transformation (4.65) we calculate $\rho_{ent}(p)$ and $\rho_s(p)$ which lead us to obtain the optimal EW for $\rho_{ent}(p)$, by using the Eq.(4.70) and compute the Hilbert-Schmidt measure for the transformed state $\rho_{ent}(p)$.

For instance, we consider one of the PPT BSD entangled states given by (4.57) as

$$\rho_{ent.}^{(1,0,0,0)}(0) = \frac{1}{16} \{I \otimes I - \frac{1}{3}(\gamma^0 \otimes \gamma^0 - \gamma^1 \otimes \gamma^1 - \gamma^2 \otimes \gamma^2 - \gamma^3 \otimes \gamma^3 - \gamma^5 \otimes \gamma^5)\}. \quad (4.71)$$

Then, Eq.(4.60) implies that the optimal EW

$$W_{opt.}^{(2;4;0,1,1,1)} = I \otimes I + \gamma^0 \otimes \gamma^0 - \gamma^1 \otimes \gamma^1 - \gamma^2 \otimes \gamma^2 - \gamma^3 \otimes \gamma^3 - \gamma^5 \otimes \gamma^5, \quad (4.72)$$

detects $\rho_{ent.}^{(1,0,0,0)}(0)$. It should be noticed that the optimal EW (4.72) is a Lorentz invariant EW in the sense that

$$W_{opt.}'^{(2;4;0,1,1,1)} = [D^{-1}(L(P)) \otimes D^{-1}(L(P))] W_{opt.}^{(2;4;0,1,1,1)} [D(L(P)) \otimes D(L(P))] =$$

$$I_{16} + g_{\mu\nu} L(p)_\alpha^\mu L(p)_\beta^\nu \gamma^\alpha \otimes \gamma^\beta - [\det(L(P))]^2 \gamma^5 \otimes \gamma^5 = W_{opt.}^{(2;4;0,1,1,1)}, \quad (4.73)$$

where, we have used the fact that $D^{-1}(L(P))\gamma^\mu D(L(P)) = L(P)_\nu^\mu \gamma^\nu$.

Now, by using (4.69) and (4.70), one can write

$$\rho_s(0) = \rho_{ent}(0) - Tr(\rho_{ent}(0) W_{opt.}^{(2;4;0,1,1,1)}) W_{opt.}^{(2;4;0,1,1,1)} + \varepsilon(0) \mathbf{1}, \quad (4.74)$$

where,

$$\varepsilon(0) := \langle \rho_s(0), \rho_s(0) - \rho_{ent}(0) \rangle = \frac{Tr(\rho_{ent}(0) W_{opt.}^{(2;4;0,1,1,1)}) [Tr(W_{opt.}^{(2;4;0,1,1,1)2}) - 1]}{Tr(W_{opt.}^{(2;4;0,1,1,1)})}. \quad (4.75)$$

In the Eq.(4.75), we have used the optimality of $W_{opt.}^{(2;4;0,1,1,1)}$ to write $Tr(\rho_s(0) W_{opt.}^{(2;4;0,1,1,1)}) = 0$.

By substituting (4.71) and (4.72) in (4.75) and using (4.60), one can obtain $\varepsilon(0) = -\frac{95}{24}$ and then

$$\rho_s(0) = -\frac{31}{48} \{5I \otimes I - \gamma^0 \otimes \gamma^0 + \gamma^1 \otimes \gamma^1 + \gamma^2 \otimes \gamma^2 + \gamma^3 \otimes \gamma^3 + \gamma^5 \otimes \gamma^5\}. \quad (4.76)$$

The state (4.76) is clearly separable since the states $I \otimes I \pm \gamma^\mu \otimes \gamma^\mu$ are product states for $\mu = 0, 1, 2, 3, 5$. We normalize the obtained state $\rho_s(0)$ as

$$\rho_s(0) = \frac{1}{80} \{5I \otimes I - \gamma^0 \otimes \gamma^0 + \gamma^1 \otimes \gamma^1 + \gamma^2 \otimes \gamma^2 + \gamma^3 \otimes \gamma^3 + \gamma^5 \otimes \gamma^5\} \quad (4.77)$$

such that $Tr(\rho_s(0)) = 1$. By this normalization, $\varepsilon(0)$ changes to $\varepsilon(0) = -\frac{1}{120}$ and by using (4.70), $W_{opt.}^{(2;4;0,1,1,1)}$ is rewritten as

$$W_{opt.}^{(2;4;0,1,1,1)} = \frac{1}{4\sqrt{5}} (I \otimes I + \gamma^0 \otimes \gamma^0 - \gamma^1 \otimes \gamma^1 - \gamma^2 \otimes \gamma^2 - \gamma^3 \otimes \gamma^3 - \gamma^5 \otimes \gamma^5), \quad (4.78)$$

Then, by using (4.69) and (4.78), we calculate the Hilbert-Schmidt measure of $\rho_{ent}^{(1,0,0,0)}(0)$ as

$$D(\rho_{ent}^{(1,0,0,0)}(0)) = \|\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0)\| = -Tr(\rho_{ent}^{(1,0,0,0)}(0)W_{opt.}^{(2;4;0,1,1,1)}) = \frac{2}{3} \cdot \frac{1}{4\sqrt{5}} = \frac{\sqrt{5}}{30}. \quad (4.79)$$

Now, by using the Lorentz transformation (4.65), one can evaluate the transformed states

$\rho_{ent}^{(1,0,0,0)}(p)$ and $\rho_s(p)$ as

$$\begin{aligned} \rho_{ent}^{(1,0,0,0)}(p) = & \frac{\cosh^4(\xi/2)}{16 \cosh^2(\xi)} \{ (1 + \tanh^2(\xi/2))^2 I_4 \otimes I_4 + 2 \tanh(\xi/2) (1 + \tanh^2(\xi/2)) (I_2 \otimes I_2 \otimes \gamma^0 \gamma^3 + \gamma^0 \gamma^3 \otimes I_2 \otimes I_2) + \\ & 4 \tanh^2(\xi/2) \gamma^0 \gamma^3 \otimes \gamma^0 \gamma^3 + \frac{1}{3} [(1 - \tanh^2(\xi/2))^2 (-\gamma^0 \otimes \gamma^0 + \gamma^3 \otimes \gamma^3) + (1 + \tanh^2(\xi/2))^2 (\gamma^1 \otimes \gamma^1 + \gamma^2 \otimes \gamma^2 + \gamma^5 \otimes \gamma^5) - \\ & 2i \tanh(\xi/2) (1 + \tanh^2(\xi/2)) (\gamma^1 \otimes \gamma^2 \gamma^5 + \gamma^2 \gamma^5 \otimes \gamma^1 - \gamma^2 \otimes \gamma^1 \gamma^5 - \gamma^1 \gamma^5 \otimes \gamma^2 - \gamma^5 \otimes \gamma^1 \gamma^2 - \gamma^1 \gamma^2 \otimes \gamma^5) - \\ & 4 \tanh^2(\xi/2) (\gamma^1 \gamma^5 \otimes \gamma^1 \gamma^5 + \gamma^2 \gamma^5 \otimes \gamma^2 \gamma^5 + \gamma^1 \gamma^2 \otimes \gamma^1 \gamma^2)] \}, \end{aligned} \quad (4.80)$$

and

$$\begin{aligned} \rho_s(p) = & \frac{\cosh^4(\xi/2)}{80 \cosh^2(\xi)} \{ 5(1 + \tanh^2(\xi/2))^2 I_4 \otimes I_4 + 10 \tanh(\xi/2) (1 + \tanh^2(\xi/2)) (I_2 \otimes I_2 \otimes \gamma^0 \gamma^3 + \gamma^0 \gamma^3 \otimes I_2 \otimes I_2) + \\ & 20 \tanh^2(\xi/2) \gamma^0 \gamma^3 \otimes \gamma^0 \gamma^3 + (1 - \tanh^2(\xi/2))^2 (-\gamma^0 \otimes \gamma^0 + \gamma^3 \otimes \gamma^3) + (1 + \tanh^2(\xi/2))^2 (\gamma^1 \otimes \gamma^1 + \gamma^2 \otimes \gamma^2 + \gamma^5 \otimes \gamma^5) - \\ & 2i \tanh(\xi/2) (1 + \tanh^2(\xi/2)) (\gamma^1 \otimes \gamma^2 \gamma^5 + \gamma^2 \gamma^5 \otimes \gamma^1 - \gamma^2 \otimes \gamma^1 \gamma^5 - \gamma^1 \gamma^5 \otimes \gamma^2 - \gamma^5 \otimes \gamma^1 \gamma^2 - \gamma^1 \gamma^2 \otimes \gamma^5) - \\ & 4 \tanh^2(\xi/2) (\gamma^1 \gamma^5 \otimes \gamma^1 \gamma^5 + \gamma^2 \gamma^5 \otimes \gamma^2 \gamma^5 + \gamma^1 \gamma^2 \otimes \gamma^1 \gamma^2) \}, \end{aligned} \quad (4.81)$$

respectively. Then, we obtain

$$\begin{aligned} \varepsilon(p) = & \langle \rho_s(p), \rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p) \rangle = Tr[\rho_s(p)(\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p))] = \\ & - \frac{\cosh^8(\xi/2)}{600 \cosh^4(\xi)} \{ 5(1 + \tanh^8(\xi/2)) + 28(\tanh^2(\xi/2) + \tanh^6(\xi/2)) + 126 \tanh^4(\xi/2) \}. \end{aligned} \quad (4.82)$$

Then, by using (4.70), we obtain the optimal EW associated with $\rho_{ent}^{(1,0,0,0)}(p)$ as

$$\begin{aligned} W_{opt}(p) = & \frac{\cosh^4(\xi/2)}{120 \cosh^2(\xi) \|\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p)\|} \{ \frac{180 \cosh^2(\xi)}{\cosh^4(\xi/2)} \|\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p)\|^2 I \otimes I + \\ & (1 - \tanh^2(\xi/2))^2 (\gamma^0 \otimes \gamma^0 - \gamma^3 \otimes \gamma^3) - (1 + \tanh^2(\xi/2))^2 (\gamma^1 \otimes \gamma^1 + \gamma^2 \otimes \gamma^2 + \gamma^5 \otimes \gamma^5) - \\ & 2i \tanh(\xi/2) (1 + \tanh^2(\xi/2))^2 (-\gamma^1 \otimes \gamma^2 \gamma^5 - \gamma^2 \gamma^5 \otimes \gamma^1 + \gamma^2 \otimes \gamma^1 \gamma^5 + \gamma^1 \gamma^5 \otimes \gamma^2 + \gamma^5 \otimes \gamma^1 \gamma^2 + \gamma^1 \gamma^2 \otimes \gamma^5) + \end{aligned}$$

$$4 \tanh^2(\xi/2)(\gamma^1\gamma^5 \otimes \gamma^1\gamma^5 + \gamma^2\gamma^5 \otimes \gamma^2\gamma^5 + \gamma^1\gamma^2 \otimes \gamma^1\gamma^2)\}. \quad (4.83)$$

In the appendix E, we show that $W_{opt}(p)$ in (4.83) is an EW. Now, we evaluate the Hilbert-Schmidt measure of $\rho_{ent}^{(1,0,0,0)}(p)$ as follows

$$D(\rho_{ent}^{(1,0,0,0)}(p)) = \|\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p)\| = -Tr(W_{opt}(p)\rho_{ent}^{(1,0,0,0)}(p)) = \frac{\cosh^4(\xi/2)}{30 \cosh^2(\xi)} \left\{ \sqrt{5(1 + \tanh^8(\xi/2)) + 28(\tanh^2(\xi/2) + \tanh^6(\xi/2)) + 126 \tanh^4(\xi/2)} \right\}. \quad (4.84)$$

The above result indicates that the Hilbert-Schmidt measure of $\rho_{ent}^{(1,0,0,0)}(p)$ is larger than $\frac{\sqrt{5}}{30}$ which is the same as the Hilbert-Schmidt measure of $\rho_{ent}^{(1,0,0,0)}(0)$, i.e., $D(\rho_{ent}^{(1,0,0,0)}(p)) \geq D(\rho_{ent}^{(1,0,0,0)}(0))$. Therefore, as far as the spin quantum correlations is concerned, the amount of entanglement is not a relativistic scalar and has no invariant meaning. This result can be compared with the result of Peres, et. al. in Ref. [21], where it has been shown that the entropy of the reduced density matrix describing just the spin of a particle (without the momentum) is not Lorentz invariant. In fact, the result (4.84) indicates that the minimum value of the amount of spin entanglement of a spin entangled BSD density matrix is archived in the rest frame.

Now, we return to the rest frame and discuss the BSD density matrices which can be detected via the optimal EWs of the second kind (with $j = 1$). The case of the moving frame can be considered similar to the above discussions for the EWs of the first kind. In the rest frame, the optimal EWs of the second kind and the BSD density matrices are defined as

$$W_{opt.}^{(2;4;i_1,i_2;1)} = I_{16} + (-1)^{i_1} A'_1 \otimes A'_1 + (-1)^{i_2} A'_3 \otimes A'_3 + (-1)^{i_1+i_2} A'_5 \otimes A'_5,$$

$$\rho'_{BSD} = \frac{1}{16} I_{16} + b'_1 A'_1 \otimes A'_1 + b'_2 A'_2 \otimes A'_2 + b'_3 A'_3 \otimes A'_3 + b'_4 A'_4 \otimes A'_4 + b'_5 A'_5 \otimes A'_5 + b'_6 A'_6 \otimes A'_6$$

respectively, where $A'_1 = i\gamma^1\gamma^2$, $A'_2 = \gamma^5$, $A'_3 = \gamma^1\gamma^5$, $A'_4 = -i\gamma^2$, $A'_5 = \gamma^2\gamma^5$ and $A'_6 = -i\gamma^1$.

The positivity of ρ'_{BSD} implies that

$$\frac{1}{16} + (-1)^{i_1} b'_1 + (-1)^{i_2} b'_2 + (-1)^{i_3} b'_3 + (-1)^{i_1+i_2+i_3} b'_4 - (-1)^{i_1+i_3} b'_5 - (-1)^{i_2+i_3} b'_6 \geq 0 \quad (4.85)$$

These inequalities define a tetrahedron in the 6-dimensional space with coordinates (b'_1, \dots, b'_6) where its vertices are $\frac{1}{16}((-1)^{i_1}, 0, (-1)^{i_2}, 0, -(-1)^{i_1+i_2}, 0)$ with $i_1, i_2 \in \{0, 1\}$. In order to determine the region of entangled states, we consider the constraints defined by

$$Tr(\rho'_{BSD} W_{opt.}^{(2;4;i_1,i_2;1)}) = 16(\frac{1}{16} + (-1)^{i_1}b'_1 + (-1)^{i_2}b'_3 + (-1)^{i_1+i_2}b'_5) \geq 0. \quad (4.86)$$

The inequalities (4.85) and (4.86) form a polytope with vertices $\frac{1}{16}(\pm 1, 0, 0, 0, 0, 0)$, $\frac{1}{16}((-1)^{i_1}, 0, 0, (-1)^{i_2}, 0, -(-1)^{i_1+i_2})$, $\frac{1}{16}(0, (-1)^{i_1}, (-1)^{i_2}, 0, 0, -(-1)^{i_1+i_2})$ and $\frac{1}{16}(0, (-1)^{i_1}, 0, (-1)^{i_2}, -(-1)^{i_1+i_2}, 0)$. Therefore, the polytope defined by inequalities of (4.85) is divided to five regions: the central region which corresponds to the octahedron, is the region of separable states. The other four regions which are all equivalent are in fact the smaller tetrahedrons which are associated with the entangled states. Each of these tetrahedrons corresponds to an offence of one of the inequalities in (4.86).

4.2 Non-relativistic entangled states which can be detected by $W_{opt.}^{(m;d;i_1,\dots,i_d)}$

Now, we consider the multispinor systems with density matrices of the form

$$\rho_{BSD}^{(m;d)} := b_0 I_{2^{md/2}} + \sum_{\mu=1}^{2^d-1} b_\mu \underbrace{A_\mu \otimes \dots \otimes A_\mu}_m \quad (4.87)$$

as a generalization of BSD density matrices to the cases of multipartite and higher dimensional systems, where A_μ 's are hermitian operators obtained by all possible multiplications of $\gamma_i^{(d)}$, $i = 1, \dots, 2d$ as before. The determination of the region of PPT entangled states detectable by $W_{opt.}^{(m;d;i_1,\dots,i_d)}$ is similar to the case of the bipartite four-dimensional space-time.

We consider the following particular density matrices

$$\rho_{BSD}^{(m;d)} := b_0 I_{2^{md/2}} + \sum_{i=1}^{d+1} b_i \underbrace{\gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}}_m \quad (4.88)$$

Due to tracelessness of $\gamma_i^{(d)}$, the condition $Tr(\rho_{BSD}^{(m;d)}) = 1$ gives $b_0 = \frac{1}{2^{md/2}}$ and the positivity of

$\rho_{BSD}^{(m;d)}$ imposes the constraints

$$\frac{1}{2^{md/2}} + \sum_{k=1}^d (-1)^{i_k} b_k + (-i)^{md/2} (-1)^{i_1+\dots+i_d} b_{d+1} \geq 0 \quad , \quad \forall (i_1, i_2, \dots, i_d) \in \{0, 1\}^d \quad (4.89)$$

to its eigenvalues. The intersection of these 2^d halfspaces form a simplex polygon in a $d+1$ dimensional space with coordinate variables b_i (excepted b_0). Furthermore if we want $\rho_{BSD}^{(m;d)}$ becomes a PPT entangled state in the sense that its partial transpose is positive definite with respect to each subsystem, then we will obtain additional constraints which must be satisfied. For instance, the positivity of the partial transpose with respect to any particle, i.e., $\rho_{BSD}^{(m;d)T_i} \geq 0$, $i = 1, 2, \dots, m$ imposes the following constraints

$$\frac{1}{2^{md/2}} + \sum_{k=0}^{d/2-1} (-1)^{i_{2k+1}} b_{2k+1} - \sum_{k=1}^{d/2} (-1)^{i_{2k}} b_{2k} + (-i)^{md/2} (-1)^{i_1+\dots+i_d} b_{d+1} \geq 0 \quad , \forall (i_1, i_2, \dots, i_d) \in \{0, 1\}^d$$

where, we have used the fact that all γ matrices with odd index are symmetric and all matrices with even index are antisymmetric (see appendix A). In order to determine the region of PPT entangled states, we consider the constraints obtained by

$$Tr(\rho_{BSD}^{(m;d)} W_{opt.}^{(m;d;i_1,\dots,i_d)}) = 1 + 2^{md/2} \left(\sum_{k=1}^d (-1)^{i_k} b_k - (-i)^{md/2} (-1)^{i_1+\dots+i_d} b_{d+1} \right) \geq 0. \quad (4.90)$$

The inequalities (4.89) and (4.90) form a polyhedron with vertices $(\pm 1, 0, \dots, 0)$, $(0, \pm 1, 0, \dots, 0)$, ..., $(0, \dots, 0, \pm 1)$ which are the same as the vertex points of the feasible region. Therefore, the polyhedron defined by inequalities of (4.89) is divided to $2^d + 1$ regions: the central region which is defined by (4.89) and (4.90), corresponds to the region of separable states. The other 2^d regions are in fact the smaller polyhedrons which are associated with the PPT entangled states. Each of these polyhedrons corresponds to an offence of one of the inequalities in (4.90).

We note that, in the the case of $d = m = 2$ with

$$W_{opt.}^{(2;2;i_1,i_2)} = I_4 + (-1)^{i_1} \sigma_x \otimes \sigma_x + (-1)^{i_2} \sigma_y \otimes \sigma_y + (-1)^{i_1+i_2} \sigma_z \otimes \sigma_z,$$

$$\rho_{BSD}^{(2;2)} = \frac{1}{4} I_4 + b_1 \sigma_x \otimes \sigma_x + b_2 \sigma_y \otimes \sigma_y + b_3 \sigma_z \otimes \sigma_z,$$

the Eq.(4.89) implies that

$$\frac{1}{4} + (-1)^{i_1}b_1 + (-1)^{i_2}b_2 - (-1)^{i_1+i_2}b_3 \geq 0. \quad (4.91)$$

These inequalities define a tetrahedron in the 3-dimensional space with coordinates (b_1, b_2, b_3) where its vertices are $(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$, $(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4})$, $(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4})$ and $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. In order to determine the region of entangled states, we consider the constraints obtained by

$$Tr(\rho_{BSD}^{(2;2)} W_{opt.}^{(2;2;i_1,i_2)}) = 4(\frac{1}{4} + (-1)^{i_1}b_1 + (-1)^{i_2}b_2 - (-1)^{1+i_1+i_2}b_3) \geq 0. \quad (4.92)$$

The inequalities (4.91) and (4.92) form an octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Therefore, the tetrahedron defined by inequalities of (4.91) is divided to five regions: the central region which corresponds to the octahedron, is the region of separable states. The other four regions which are all equivalent are in fact the smaller tetrahedrons which are associated with the PPT entangled states. Each of these tetrahedrons corresponds to an offence of one of the inequalities in (4.92). It is interesting to note that, in this case the nonnegativity of $Tr(\rho_{BSD}^{(2;2)} W_{opt.}^{(2;2;i_1,i_2)}) < 0$ is equivalent to $C(\rho_{BSD}^{(2;2)}) > 0$, where C is the concurrence introduced by Wootters [55].

4.3 Non-relativistic entangled states which can be detected by $W_{opt.}^{I(m;d;i_1,i_2;j)}$

Now we assert that $W_{opt.}^{I(m;d;i_1,i_2;j)}$ given by (3.35) can also detect some entangled multispinor mixed density matrices. To this aim we consider only the following density matrices

$$\rho_{BSD}^{I(m;d)} := b'_0 I_{2^{md/2}} + \sum_{i=1}^{3d/2} b'_i \underbrace{A'_i \otimes A'_i \otimes \dots \otimes A'_i}_m, \quad (4.93)$$

with $b'_0 = \frac{1}{2^{md/2}}$, where the general density matrices as in (4.87) can be considered similarly.

The positivity of density matrix $\rho_{BSD}^{I(m;d)}$ imposes

$$b'_0 + \sum_{k=1}^{d/2+1} (-1)^{i_k} b'_k + \sum_{k=2}^{d/2} (-1)^{i_1+i_{d/2+1}+i_k} b'_{d/2+k} + \sum_{k=1}^{d/2} (-1)^{m/2+i_{d/2+1}+i_k} b'_{d+k} \geq 0 \quad (4.94)$$

with $i_1, \dots, i_{d/2+1} \in \{0, 1\}$, to its eigenvalues. The intersection of these halfspaces form a simplex polygon in a $3d/2$ dimensional space with coordinate variables b'_i , $i = 1, \dots, 3d/2$ (excepted b_0).

The condition for detectability of $\rho'_{BSD}^{(m;d)}$ by $W'_{opt.}{}^{(m;d;i_1,i_2;j)}$ can be written as

$$Tr(\rho'_{BSD}{}^{(m;d)} W'_{opt.}{}^{(m;d;i_1,i_2;j)}) = 1 + 2^{md/2}((-1)^{i_1}b'_j + (-1)^{i_2}b'_{d/2+j} - (-1)^{m/2+i_1+i_2}b'_{d+j}) < 0. \quad (4.95)$$

As we will show in the following section, the EWs $W'_{opt.}{}^{(m;d;i_1,i_2;j)}$ are decomposable EWs and so can not detect PPT entangled states.

5 Decomposability or non-decomposability of BSD multispinor EWs

Another interesting feature of EWs is their decomposability or non-decomposability. Clearly d-EW can not detect PPT entangled states (these states are also called bound entangled states because they have the peculiar property that no entanglement can be distilled from them by local operations [56]) whereas there are some bound entangled states which can be detected by a nd-EW. In the previous section, it was shown that there exist some bound entangled states which can be detected by the optimal EWs of the first kind, whereas the EWs of the second kind can not detect bound entangled states. In fact, the detectability or non-detectability of bound entangled states is due to non-decomposability or decomposability of the corresponding EWs where in the following, we discuss this particular property of the optimal EWs of the first and second kinds.

5.1 The region of non-decomposable EWs of the first kind

First consider the first kind of BSD multispinor EWs $W^{(m;d)}$. The inequalities of Eq.(3.22) show that in the space of parameters a_i , all of these EWs lie inside the hypercube (by fixing

a_0). The operator $W^{(m;d)}$ is positive in the region defined by the following inequalities

$$a_0 + \sum_{k=1}^d (-1)^{i_k} a_k + (-i)^{md/2} (-1)^{i_1+\dots+i_d} a_{d+1} \geq 0 \quad (i_1, \dots, i_d) \in \{0, 1\}^d. \quad (5.96)$$

Now, consider the 2^d coordinates $(a_1, \dots, a_d, a_{d+1}) \in \{((-1)^{i_1}, \dots, (-1)^{i_d}, -(-i)^{md/2}(-1)^{i_1+\dots+i_d}) : (i_1, \dots, i_d) \in \{0, 1\}^d\}$. These coordinates correspond to the optimal EWs given by Eq.(3.24).

The partial transpositions of the optimal EWs can be written as

$$W_{opt.}^{(m;d;i_1,\dots,i_d)T_i} = I - \sum_{j=1}^d (-1)^{i_{2j}} \gamma_{2j}^{(d)} \otimes \dots \otimes \gamma_{2j}^{(d)} + \sum_{j=1}^d (-1)^{i_{2j-1}} \gamma_{2j-1}^{(d)} \otimes \dots \otimes \gamma_{2j-1}^{(d)} - (-i)^{md/2} (-1)^{i_1+\dots+i_d} \gamma_{d+1}^{(d)} \otimes \dots \otimes \gamma_{d+1}^{(d)}, \quad (5.97)$$

for $i = 1, 2, \dots, m$. Note that, $W_{opt.}^{(m;d;i_1,\dots,i_d)T_i}$ are not positive for $d \neq 2$. In fact, the eigenvalues of $W_{opt.}^{(m;d;i_1,\dots,i_d)T_i}$ are given by

$$\lambda_{k_1,\dots,k_d}^{(m;d;i_1,\dots,i_d)} = 1 - \sum_{j=1}^{d/2} \sum_{l=1}^{d/2} (-1)^{i_{2j}} (-1)^{k_{2l}} + \sum_{j=1}^{d/2} \sum_{l=1}^{d/2} (-1)^{i_{2j-1}} (-1)^{k_{2l-1}} - (-i)^{md} (-1)^{i_1+\dots+i_d+k_1+\dots+k_d} \quad (5.98)$$

which are not positive with respect to none of the particles. For example, consider the case $i_1 = i_2 = \dots = i_d = 0$. Then the eigenvalues will be

$$\lambda_{k_1,\dots,k_d}^{(m;d;0,\dots,0)} = 1 - \sum_{l=1}^{d/2} (-1)^{k_{2l}} + \sum_{l=1}^{d/2} (-1)^{k_{2l-1}} - (-1)^{k_1+\dots+k_d}, \quad (5.99)$$

where, the most negative eigenvalue is obtained by taking $k_{2l} = 0, k_{2l-1} = 1, l = 1, 2, \dots, d/2$, i.e., we have

$$\lambda_{k_{2l}=0, k_{2l-1}=1}^{(m;d;0,\dots,0)} = 1 - d - (-1)^{d/2} \leq 2 - d < 0, \quad \text{for } d > 2. \quad (5.100)$$

For $d = 2$, the EW is the same as the EW of the second kind and we have

$$W_{opt.}^{(m;2;i_1,i_2)T_i} = I + (-1)^{i_1} \gamma_1 \otimes \dots \otimes \gamma_1 - (-1)^{i_2} \gamma_2 \otimes \dots \otimes \gamma_2 - (-1)^{m/2+i_1+i_2} \gamma_3 \otimes \dots \otimes \gamma_3, \quad (5.101)$$

which is positive for all values of m and $i_1, i_2 \in \{0, 1\}$.

We discuss the non-decomposability of EWs only in the case of $d = 4, m = 2$, the multipartite case can be discussed similarly. Now we consider the vertices of the density matrices'

region given by (4.57) (recall that, all of these density matrices are PPT). In order to determine the region of non-decomposable EWs, we take the constraints obtained by

$$\text{Tr}(\rho_{i_1, \dots, i_4}^{(2;4)} W^{(2;4)}) = 16(1 - \frac{1}{3}[(-1)^{i_1}a_1 + (-1)^{i_2}a_2 + (-1)^{i_3}a_3 + (-1)^{i_4}a_4 - (-1)^{i_1+\dots+i_4}a_5]) < 0 \quad (5.102)$$

which is equivalent to

$$(-1)^{i_1}a_1 + (-1)^{i_2}a_2 + (-1)^{i_3}a_3 + (-1)^{i_4}a_4 - (-1)^{i_1+\dots+i_4}a_5 > 3. \quad (5.103)$$

It can be seen that the minimum value of $1 - \frac{1}{3}[(-1)^{i_1}a_1 + (-1)^{i_2}a_2 + (-1)^{i_3}a_3 + (-1)^{i_4}a_4 - (-1)^{i_1+\dots+i_4}a_5]$ is obtained by choosing the parameters (a_1, \dots, a_5) as $((-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}, (-1)^{i_4}, -(-1)^{i_1+\dots+i_4})$, which are the same as the optimal EWs, i.e., all of the optimal EWs $W_{opt}^{(2;4;i_1, \dots, i_4)}$ are non-decomposable. Then, we will have

$$\min_{a_1, \dots, a_5} (1 - \frac{1}{3}[(-1)^{i_1}a_1 + (-1)^{i_2}a_2 + (-1)^{i_3}a_3 + (-1)^{i_4}a_4 - (-1)^{i_1+\dots+i_4}a_5]) = 1 - \frac{5}{3} = -\frac{2}{3}. \quad (5.104)$$

In fact, the EWs $W^{(2;4)}$ satisfying the inequalities (5.103) are non-decomposable EWs.

5.2 Decomposability of EWs of the second kind

Now, consider the second kind of BSD multispinor EWs. In the space of parameters a'_i (again by fixing a'_0), all of these EWs lie inside the region defined by Eq. (3.33). The region

$$a'_0 + \sum_{k=1}^{d/2+1} (-1)^{i_k} a'_k + \sum_{k=2}^{d/2} (-1)^{i_1+i_{d/2+1}+i_k} a'_{d/2+k} + \sum_{k=1}^{d/2} (-1)^{m/2+i_{d/2+1}+i_k} a'_{d+k} \geq 0 \quad (i_1, \dots, i_d) \in \{0, 1\}^d \quad (5.105)$$

is the place where the EW is positive. Now consider the coordinates $(a'_1, \dots, a'_d, a'_{d+1}) \in ((-1)^{i_1}, 0, \dots, 0, (-1)^{i_2}, 0, \dots, 0, (-1)^{i_1+i_2})$. These parameters correspond to the optimal EWs given by (3.35) for $j = 1$ (the discussions for $j = 2, \dots, d/2$ are similar). The partial transpose

of these optimal EWs with respect to each particle is given by

$$W_{opt.}^{\prime(m;d;i_1,i_2;1)T_i} = I + (-1)^{i_1} A'_1 \otimes \dots \otimes A'_1 + (-1)^{i_2} A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1} + (-1)^{m/2+i_1+i_2} A'_{d+1} \otimes \dots \otimes A'_{d+1}, \quad (5.106)$$

Then, the eigenvalues of $W_{opt.}^{\prime(m;d;i_1,i_2;1)T_i}$ are

$$\lambda_{k_1,k_2}^{\prime(m;d;i_1,i_2)} = 1 + (-1)^{i_1+k_1} + (-1)^{i_2+k_2} + (-1)^{i_1+i_2+k_1+k_2}, \quad (5.107)$$

where, we have used the fact that $A'_{d+1} = iA'_{d/2+1}A'_1$. Then, one can easily check that $\lambda_{k_1,k_2}^{\prime(m;d;i_1,i_2)}$ are positive for all values of $i_1, i_2, k_1, k_2 \in \{0, 1\}$. Therefore, the EWs defined by (3.35) have positive partial transpose with respect to each particle and so are optimal decomposable EWs. A cone which may be formed by connecting every four points of Eq.(3.35) to its opposite positive hyperplane in Eq.(5.105) is d-EWs. Note that the remaining operators in Eq.(3.25) coming from some points in the space of parameters are either d-EW or positive. In fact, from the convexity of the EWs' region, every EW is written as a convex combination of the decomposable optimal EWs (the vertices of the EWs' region) and so is also decomposable. Therefore we conclude that all of the multispinor EWs of the second kind are decomposable and can not detect PPT entangled states.

6 Multispinor EWs which can be manipulated approximately by LP

So far, we have considered the BSD multispinor EWs which can be constructed by the exact LP method, while in this section, we consider the EWs that can be manipulated by approximate LP which come from by adding other members of Dirac γ matrices algebra to exactly soluble multispinor EWs. In all of the multispinor EWs discussed in section 3, the boundary hyperplanes arise from the vertex points which themselves come from pure product states and the resulting inequalities did not offend against the convex hull of the vertices at all. But by

adding some terms to exactly soluble EWs, it may be happen that the feasible region be convex with curvature on some boundaries and the problem can not be solved by the exact LP method. In these cases the linear constraints no longer arise from convex hull of the vertices coming from pure product states. Hence we transform such problem to the approximate LP one. Our approach is to draw the hyperplanes tangent to feasible region and parallel to hyperplanes coming from vertices and in this way we enclose the feasible regions by such hyperplanes. It is clear that in this extension, the vertices no longer arise from pure product states.

6.1 The first kind

In the case of the first kind of BSD multispinor EWs, we add one of the multiplication of the matrices γ_i , $i = 1, 2, \dots, d+1$, say, $-i\gamma_1\gamma_2$ to (3.14) as

$$W_{ap.}^{(m;d)} = a_0 I_{2^{md/2}} + \sum_{k=1}^{d+1} a_k \gamma_k^{(d)} \otimes \dots \otimes \gamma_k^{(d)} + (-i)^m a_{d+2} \gamma_1^{(d)} \gamma_2^{(d)} \otimes \dots \otimes \gamma_1^{(d)} \gamma_2^{(d)}. \quad (6.108)$$

(the subscript ap. refers to the approximate) and try to solve it by LP method. The eigenvalues of $W_{ap.}^{(m;d)}$ are

$$\lambda_{ap.; i_1, \dots, i_d}^{(m;d)} = a_0 + \sum_{k=1}^d (-1)^{i_k} a_k + i^{-md/2} (-1)^{i_1 + \dots + i_d} a_{d+1} + (-i)^m (-1)^{i_1 + i_2} a_{d+2}, \quad \forall (i_1, i_2, \dots, i_d) \in \{0, 1\}^d$$

The coordinates of the apexes which arise from pure product states are listed in the following table

Product state	$(P_1, P_2, \dots, P_d, P_{d+1}, P_{d+2})$
$ \psi_{\pm}^{(1)}\rangle$	$(\pm 1, 0, 0, \dots, 0)$
$ \psi_{\pm}^{(2)}\rangle$	$(0, \pm 1, 0, \dots, 0)$
\vdots	\vdots
$ \psi_{\pm}^{(d+1)}\rangle$	$(0, \dots, 0, \pm 1, 0)$
$ \psi_{\pm}^{(d+2)}\rangle$	$(0, 0, \dots, 0, \pm 1),$

(6.109)

where, $|\psi_{\pm}^{(i)}\rangle$ for $i = 1, \dots, d+1$ are defined as in section 3.2.1 and $|\psi_{\pm}^{(d+2)}\rangle$ are eigenvectors of $(-i)^m \gamma_1^{(d)} \gamma_2^{(d)} \otimes \dots \otimes \gamma_1^{(d)} \gamma_2^{(d)}$ with eigenvalues ± 1 . Then, the feasible region is the intersection

of the following halfspaces

$$\sqrt{2} + \sum_{k=1}^{d+2} (-1)^{i_k} P_k \geq 0, \quad (6.110)$$

where $i_1, \dots, i_{d+2} \in \{0, 1\}$ (the proof of (6.110) is given in appendix D). The inequalities (6.110) imply that the problem does not lie in the realm of exactly soluble LP problems and we have to use approximate LP. To this end we shift aforementioned hyperplanes parallel to themselves such that they reach to maximum value $\sqrt{2}$. On the other hand the maximum shifting is where the hyperplanes become tangent to convex region coming from pure product states and in this manner we will be able to encircle the feasible region by the hyperspaces defined by (6.110).

Regarding the above considerations, the problem is reduced to

$$\begin{aligned} & \text{minimize} && a_0 + \sum_{i=1}^{d+2} a_i P_i \\ & \text{subject to} && \begin{cases} \sqrt{2} + \sum_{k=1}^{d+2} (-1)^{i_k} P_k \geq 0 \\ \forall |P_k| \leq 1, \end{cases} \end{aligned} \quad (6.111)$$

for all $i_1, \dots, i_{d+2} \in \{0, 1\}$, where it can be solved by simplex method, since the intersections of the hyperspaces in (6.111) form a convex polytope.

By substitution of extreme points of the feasible region (we note that these points do not arise from pure product states), we get the approximate region of SSNNEV as intersection of the following halfspaces

$$|a_i| \leq \frac{1}{\sqrt{2}} a_0, \quad i = 1, \dots, d+1, d+2. \quad (6.112)$$

In fact, the approximated region of EWs is the complement of the region defined by $\lambda_{ap.;i_1,\dots,i_d}^{(m;d)} \geq 0$ in the hypercube defined by (6.112).

6.1.1 The region of non-decomposable (approximate) EWs of the first kind

The inequalities of Eq.(6.112) show that in the space of parameters a_i , all of the EWs $W_{ap.}^{(m;d)}$ lie inside a hypercube (by fixing a_0). Also, these EWs are positive in the region defined by the

following inequalities

$$a_0 + \sum_{k=1}^d (-1)^{i_k} a_k + (-i)^{md/2} (-1)^{i_1+\dots+i_d} a_{d+1} + (-i)^m (-1)^{i_1+i_2} a_{d+2} \geq 0 \quad (i_1, \dots, i_d) \in \{0, 1\}^d. \quad (6.113)$$

Now consider the coordinates $(a_1, \dots, a_d, a_{d+1}, a_{d+2}) \in \{((-1)^{i_1}, (-1)^{i_2}, 0, 0, \dots, 0, (-1)^{i_1+i_2}) : i_1, i_2 \in \{0, 1\}\}$. Substituting these 2^d points in $W_{ap.}^{(m;d)}$ gives the optimal EWs in the approximated region as follows

$$W_{ap.,opt.}^{(m;d;i_1,\dots,i_d)} = I + (-1)^{i_1} \gamma_1^{(d)} \otimes \dots \otimes \gamma_1^{(d)} + (-1)^{i_2} \gamma_2^{(d)} \otimes \dots \otimes \gamma_2^{(d)} + (-1)^{m/2+i_1+i_2} \gamma_1^{(d)} \gamma_2^{(d)} \otimes \dots \otimes \gamma_1^{(d)} \gamma_2^{(d)}. \quad (6.114)$$

The partial transpositions of the optimal EWs are as follows

$$W_{ap.,opt.}^{(m;d;i_1,\dots,i_d)T_i} = I + (-1)^{i_1} \gamma_1^{(d)} \otimes \dots \otimes \gamma_1^{(d)} - (-1)^{i_2} \gamma_2^{(d)} \otimes \dots \otimes \gamma_2^{(d)} + (-1)^{m/2+i_1+i_2} \gamma_1^{(d)} \gamma_2^{(d)} \otimes \dots \otimes \gamma_1^{(d)} \gamma_2^{(d)}. \quad (6.115)$$

Then, the eigenvalues of $W_{ap.,opt.}^{(m;d;i_1,\dots,i_d)T_i}$ are given by

$$\lambda_{k_1,k_2}^{(m;d;i_1,i_2)} = 1 + (-1)^{i_1+k_1} - (-1)^{i_2+k_2} + (-1)^{i_1+i_2+k_1+k_2} \quad (6.116)$$

which are not positive with respect to none of the particles. For example, in the case of $i_1 = i_2 = 0$ the eigenvalues read

$$\lambda_{k_1,k_2}^{(m;d;0,0)} = 1 + (-1)^{k_1} - (-1)^{k_2} + (-1)^{k_1+k_2} \quad (6.117)$$

where, the most negative eigenvalue is -2 which is obtained by taking $k_1 = 1, k_2 = 0$. As before, we consider the density matrices of the form

$$\rho^{(m;d)} = b_0 I_{2^{md/2}} + \sum_{k=1}^{d+1} b_k \gamma_k^{(d)} \otimes \dots \otimes \gamma_k^{(d)} + (-i)^m b_{d+2} \gamma_1^{(d)} \gamma_2^{(d)} \otimes \dots \otimes \gamma_1^{(d)} \gamma_2^{(d)} \quad (6.118)$$

Then, the positivity of $\rho^{(m;d)}$ implies that $b_0 + \sum_{k=1}^d (-1)^{i_k} b_k + (-i)^{md/2} (-1)^{i_1+\dots+i_d} b_{d+1} + (-i)^m (-1)^{i_1+i_2} b_{d+2} \geq 0$. We discuss the non-decomposability of $W_{ap.}^{(m;d)}$ only for the case of $m = 2$ and $d = 4$, the general cases can be discussed similarly. In this case, we have

$$\rho^{(2;4)} = \frac{1}{16} I_{16} + \sum_{k=1}^5 b_k \gamma_k^{(4)} \otimes \gamma_k^{(4)} - b_6 \gamma_1^{(4)} \gamma_2^{(4)} \otimes \gamma_1^{(4)} \gamma_2^{(4)} \quad (6.119)$$

Then, the vertices of the PPT density matrices' region (the region defined by the positivity conditions $\rho^{(2;4)} \geq 0$ and $\rho^{(2;4)T_i} \geq 0$ for $i = 1, 2$ which are equivalent to the inequalities $\frac{1}{16} + \sum_{k=1}^4 (-1)^{i_k} b_k + (-1)^{i_1+\dots+i_4} b_5 - (-1)^{i_1+i_2} b_6 \geq 0$) are given by

$$\rho_{i_1, i_2}^{(2;4)} = \frac{1}{16}((-1)^{i_1}, (-1)^{i_2}, 0, 0, 0, -(-1)^{i_1+i_2}), \quad i_1, i_2 \in \{0, 1\}. \quad (6.120)$$

In order to determine the region of non-decomposable EWs, we consider the constraints obtained by

$$\text{Tr}(W_{ap.}^{(2;4)} \rho_{i_1, i_2}^{(2;4)}) = 16(1 - [(-1)^{i_1} a_1 + (-1)^{i_2} a_2 - (-1)^{i_1+i_2} a_6]) < 0 \quad (6.121)$$

which are equivalent to $(-1)^{i_1} a_1 + (-1)^{i_2} a_2 - (-1)^{i_1+i_2} a_6 > 1$. It can be seen that the minimum value of $1 - [(-1)^{i_1} a_1 + (-1)^{i_2} a_2 - (-1)^{i_1+i_2} a_6]$ is obtained by choosing the parameters (a_1, \dots, a_6) as $((-1)^{i_1}, (-1)^{i_2}, 0, 0, 0, -(-1)^{i_1+i_2})$, which are the same as the optimal EWs given by (6.114), i.e., all of the optimal EWs $W_{ap., opt.}^{(2;4; i_1, i_2)}$ are non-decomposable. Then, we will have

$$\min_{a_1, a_2, a_6} (1 - [(-1)^{i_1} a_1 + (-1)^{i_2} a_2 - (-1)^{i_1+i_2} a_6]) = -2. \quad (6.122)$$

In fact, the EWs $W_{ap.}^{(2;4)}$ satisfying the inequalities (6.121) are non-decomposable approximate EWs.

6.2 The second kind

For the second kind of BSD multispinor EWs we add one of the multiplications of the matrices A'_i , $i = 1, 2, \dots, 3d/2$, say, $A'_1 A'_2$ to (3.25) as

$$W_{ap.}^{\prime(m; d)} = a'_0 I_{2^{md/2}} + \sum_{k=1}^{3d/2} a'_k A'_k \otimes \dots \otimes A'_k + a'_{3d/2+1} A'_1 A'_2 \otimes \dots \otimes A'_1 A'_2. \quad (6.123)$$

and try to solve it by LP method. The eigenvalues of $W_{ap.}^{\prime(m; d)}$ are given by

$$a'_0 + \sum_{k=1}^{d/2+1} (-1)^{i_k} a'_k + \sum_{k=2}^{d/2} (-1)^{i_1+i_{d/2+1}+i_k} a'_{d/2+k} + \sum_{k=1}^{d/2} (-1)^{m/2+i_{d/2+1}+i_k} a'_{d+k} + (-1)^{i_1+i_2} a'_{3d/2+1}, \quad (6.124)$$

for all $i_1, \dots, i_{d/2+1} \in \{0, 1\}$. The coordinates of the vertex points which arise from pure product states are listed in the following table

Product state	$(P'_1, \dots, P'_{d/2}; P'_{d/2+1}, \dots, P'_d; P'_{d+1}, \dots, P'_{3d/2}; P'_{3d/2+1})$
$ \psi_{\pm}^{(1;1)}\rangle$	$(\pm 1, 1, 1, \dots, 1; 0, 0, \dots, 0; 0, 0, \dots, 0; 0)$
\vdots	\vdots
$ \psi_{\pm}^{(1;d/2)}\rangle$	$(1, 1, \dots, 1, 1, \pm 1; 0, 0, \dots, 0; 0, \dots, 0; 0)$
$ \psi_{\pm}^{(2;d/2+1)}\rangle$	$(0, 0, \dots, 0; \pm 1, 1, \dots, 1, 1; 0, 0, \dots, 0; 0)$
\vdots	\vdots
$ \psi_{\pm}^{(2;d)}\rangle$	$(0, 0, \dots, 0; 1, 1, \dots, 1, \pm 1; 0, 0, \dots, 0; 0)$
$ \psi_{\pm}^{(3;d+1)}\rangle$	$(0, 0, \dots, 0, 0; 0, \dots, 0, 0; \pm 1, 1, \dots, 1; 0)$
\vdots	\vdots
$ \psi_{\pm}^{(3;3d/2)}\rangle$	$(0, 0, \dots, 0; 0, 0, \dots, 0; 1, \dots, 1, 1, \pm 1; 0)$
$ \psi_{\pm}^{(3;3d/2+1)}\rangle$	$(0, 0, \dots, 0, 0; 0, 0, \dots, 0; 0, 0, \dots, 0; \pm 1)$

(6.125)

where, $|\psi_{\pm}^{(i;k)}\rangle$ are common eigenvectors of the elements of the commuting set C_i and $|\psi_{\pm}^{(3;3d/2+1)}\rangle$ is an eigenvector of $A'_1 A'_2 \otimes \dots \otimes A'_1 A'_2$.

Now, according to the apexes given by (6.125), one can obtain the following inequalities

$$2 + (-1)^{i_1} P'_j + (-1)^{i_2} P'_{j+d/2} + (-1)^{i_3} P'_{j+d} + (-1)^{i_4} P'_{3d/2+1} \geq 0, \quad (6.126)$$

where $i_1, \dots, i_4 \in \{0, 1\}$ and $j \in \{1, \dots, d/2\}$ (the proof is given in appendix D). Therefore, the approximated feasible region is the intersection of the halfspaces defined by (6.126). The hyperplanes surrounding the feasible region are given by

$$(-1)^{i_1} P'_j + (-1)^{i_2} P'_{j+d/2} + (-1)^{i_3} P'_{j+d} + (-1)^{i_4} P'_{3d/2+1} = 2. \quad (6.127)$$

The inequalities (6.126) imply that the problem does not lie in the realm of exactly soluble LP problems and we have to use approximate LP. To this aim we shift aforementioned hyperplanes parallel to themselves such that they reach to maximum value 2. On the other hand the

maximum shifting is where the hyperplanes (6.127) become tangent to convex region coming from pure product states and in this manner we will be able to encircle the feasible region by the hyperplanes defined by (6.127).

Regarding the inequalities (6.126), the manipulation of EWs is reduced to the following approximate LP

$$\begin{aligned} & \text{minimize} && a'_0 + \sum_{i=1}^{3d/2+1} a'_i P'_i \\ & \text{subject to} && \begin{cases} 2 + (-1)^{i_1} P'_j + (-1)^{i_2} P'_{j+d/2} + (-1)^{i_3} P'_{j+d} + (-1)^{i_4} P'_{3d/2+1} \geq 0 \\ \forall |P_k| \leq 1, \end{cases} \end{aligned} \quad (6.128)$$

for all $i_1, \dots, i_4 \in \{0, 1\}$ and $j \in \{1, \dots, d/2\}$, where it can be solved by simplex method.

In order that the expectation value of $W'_{ap.}(m;d)$ over all separable states be positive, the following constraints must be fulfilled

$$|a'_i| \leq \frac{1}{2}a'_0, \quad i = 1, \dots, 3d/2 + 1. \quad (6.129)$$

6.2.1 The region of non-decomposable (approximate) EWs of the second kind

The region defined by

$$a'_0 + \sum_{k=1}^{d/2+1} (-1)^{i_k} a'_k + \sum_{k=2}^{d/2} (-1)^{i_1+i_{d/2+1}+i_k} a'_{d/2+k} + \sum_{k=1}^{d/2} (-1)^{m/2+i_{d/2+1}+i_k} a'_{d+k} + (-1)^{i_1+i_2} a'_{3d/2+1} \geq 0, \quad (6.130)$$

for $i_1, \dots, i_{d/2+1}; i_{3d/2+1} \in \{0, 1\}$ is the region where $W'_{ap.}(m;d)$ is positive.

From (6.126), it can be seen that the optimal EWs in the approximated region are given by

$$\begin{aligned} W'_{ap.,opt.}(m;d;i_1,\dots,i_{d/2+1}) &= I_{2^{md/2}} + \sum_{k=1}^{d/2+1} (-1)^{i_k} A'_k \otimes \dots \otimes A'_k + \sum_{k=2}^{d/2} (-1)^{i_1+i_{d/2+1}+i_k} A'_{d/2+k} \otimes \dots \otimes A'_{d/2+k} + \\ &\quad \sum_{k=1}^{d/2} (-1)^{m/2+i_{d/2+1}+i_k} A'_{d+k} \otimes \dots \otimes A'_{d+k} + (-1)^{i_1+i_2} A'_1 A'_2 \otimes \dots \otimes A'_1 A'_2. \end{aligned} \quad (6.131)$$

The partial transpositions of optimal EWs $W'_{ap.,opt.}(m;d;i_1,\dots,i_{d/2+1})$ for $m = 2, d = 4$ are given by

$$W'^{(2;4;i_1,i_2,i_3)T_i}_{ap.,opt.} = I_{16} + (-1)^{i_1} A'_1 \otimes A'_1 + (-1)^{i_2} A'_2 \otimes A'_2 + (-1)^{i_3} A'_3 \otimes A'_3 + (-1)^{i_1+i_2+i_3} A'_4 \otimes A'_4 +$$

$$(-1)^{i_1+i_3} A'_5 \otimes A'_5 + (-1)^{i_2+i_3} A'_6 \otimes A'_6 + (-1)^{i_1+i_2} A'_1 A'_2 \otimes A'_1 A'_2. \quad (6.132)$$

The eigenvalues of $W'_{ap.,opt.}(2;4;i_1,i_2,i_3)T_i$ are given by

$$\begin{aligned} \lambda'_{k_1,k_2,k_3}(2;4;i_1,i_2,i_3) &= 1 + (-1)^{i_1+k_1} + (-1)^{i_2+k_2} + (-1)^{i_3+k_3} + (-1)^{i_1+i_2+i_3+k_1+k_2+k_3} - \\ &\quad (-1)^{i_1+i_3+k_1+k_3} - (-1)^{i_2+i_3+k_2+k_3} + (-1)^{i_1+i_2+k_1+k_2}. \end{aligned} \quad (6.133)$$

where, for a given $W'_{ap.,opt.}(2;4;i_1,i_2,i_3)T_i$ are not necessarily positive for all values of $k_1, k_2, k_3 \in \{0, 1\}$, for example for $W'_{ap.,opt.}(2;4;0,0,0)T_i$, the most negative eigenvalue is given by -4 which is obtained by taking $k_1 = k_2 = k_3 = 1$. This implies that the optimal EWs $W'_{ap.,opt.}(2;4;i_1,i_2,i_3)T_i$ are not necessarily decomposable. Now, we consider the density matrices of the form

$$\rho'^{(m;d)} = b'_0 I_{2^{md/2}} + \sum_{k=1}^{3d/2} b'_k A'_k \otimes \dots \otimes A'_k + b'_{3d/2+1} A'_1 A'_2 \otimes \dots \otimes A'_1 A'_2. \quad (6.134)$$

Then, for a bipartite system in the four dimensional space-time, the vertices of the PPT density matrices' region (the region defined by the positivity conditions $\rho'^{(2;4)} \geq 0$ and $\rho'^{(2;4)T_i} \geq 0$, $i = 1, 2$), are given by

$$\rho'_{i_1,i_2}(2;4) = \frac{1}{16}((-1)^{i_1}, (-1)^{i_2}, 0, 0, 0, 0, (-1)^{i_1+i_2}), \quad i_1, i_2 \in \{0, 1\}. \quad (6.135)$$

Again, by using (6.123) and (6.135), the constraints obtained by

$$Tr(\rho'_{i_1,i_2}(2;4) W'_{ap.}(2;4)) = 16(1 - [(-1)^{i_1} a'_1 + (-1)^{i_2} a'_2 + (-1)^{i_1+i_2} a'_7]) < 0 \quad (6.136)$$

which are equivalent to $(-1)^{i_1} a'_1 + (-1)^{i_2} a'_2 - (-1)^{i_1+i_2} a'_7 > 1$, partially determine the region of non-decomposable EWs in the approximated region of EWs. It can be seen that the minimum value of $1 - [(-1)^{i_1} a'_1 + (-1)^{i_2} a'_2 - (-1)^{i_1+i_2} a'_7]$ is obtained by choosing the parameters a'_1, a'_2 and a'_7 as $(-1)^{i_1}, (-1)^{i_2}$ and $-(-1)^{i_1+i_2}$, respectively. Then, we will have

$$\min_{a'_1, a'_2, a'_7} (1 - [(-1)^{i_1} a'_1 + (-1)^{i_2} a'_2 - (-1)^{i_1+i_2} a'_7]) = -2. \quad (6.137)$$

In fact, the EWs $W'_{ap.}(2;4)$ satisfying the inequalities (6.136) are non-decomposable EWs.

7 The case of odd m

In this section, we discuss the case of odd number of d -dimensional spinors, briefly. Similar to the case of even m , we need to construct EWs via hermitian commuting operators in order to calculate the corresponding eigenvalues easily. To do so, we define two kinds of operators as follows:

7.1 EWs of the first kind

In the case of odd m , we will consider the following hermitian matrix

$$W^{(m;d)} = a_0 I_{2^{md/2}} + \sum_{i=1}^{d/2} a_i \underbrace{\gamma_i^{(d)} \otimes \dots \otimes \gamma_i^{(d)}}_{m-1} \otimes A'_i + \sum_{i=1}^{d/2} a_{d/2+i} \underbrace{\gamma_{d/2+i}^{(d)} \otimes \dots \otimes \gamma_{d/2+i}^{(d)}}_{m-1} \otimes A'_i + a_{d+1} \underbrace{\gamma_{d+1}^{(d)} \otimes \dots \otimes \gamma_{d+1}^{(d)}}_{m-1} \otimes I_{2^{d/2}}, \quad (7.138)$$

where, A'_i for $i = 1, 2, \dots, d/2$ are $d/2$ commuting operators which can be taken from each of three commuting sets C_1, C_2 and C_3 defined in (3.13).

Again, in order to turn the observable (7.138) to an EW, we need to choose the parameters a_j , $j = 1, 2, \dots, d+1$ in such a way that it becomes a non-positive operator with positive expectation values in any pure product state. As in the case of even m , in this case the problem reduces to the LP one, where the feasible region, EWs' region and the region of detectable entangled states can be determined similarly.

7.2 EWs of the second kind

In the second kind, we consider the following hermitian matrix

$$W'^{(m;d)} = a'_0 I_{2^{md/2}} + \sum_{i=1}^{d/2} a'_i \underbrace{A'_i \otimes \dots \otimes A'_i}_{m-1} \otimes A'_i + \sum_{i=1}^{d/2} a'_{d/2+i} \underbrace{A'_{d/2+i} \otimes \dots \otimes A'_{d/2+i}}_{m-1} \otimes A'_i + \sum_{i=1}^{d/2} a'_{d+i} \underbrace{A'_{d+i} \otimes \dots \otimes A'_{d+i}}_{m-1} \otimes I_{2^{d/2}}, \quad (7.139)$$

where, A'_i for $i = 1, 2, \dots, d/2$ belong to the commuting set C_1 , $A'_{d/2+i}$, $i = 1, 2, \dots, d/2$ belong to the commuting set C_2 and A'_{d+i} , $i = 1, 2, \dots, d/2$ belong to the commuting set C_3 .

All of discussions about the second kind of EWs in the case of even m , can be applied in this case similarly.

8 Conclusion

Two kinds of Bell-states diagonal multispinor EWs manipulatable via the exact LP method, were constructed in order to study the entanglement properties of the relativistic and non-relativistic multispinor systems in the space-time of arbitrary dimension d , where the first kind can detect some Bell-states diagonal multispinor PPT entangled states. In particular, in the case of bipartite system in the four-dimensional space-time, the Bell-type and iso-concurrence type states were introduced and it was shown that, these states also the spinor “EPR” states which are special kinds of iso-concurrence type entangled states are detected by the constructed EWs. Moreover, it was shown that the spin entanglement of a spin entangled BSD density matrix increases under the Lorentz transformation. The decomposability or non-decomposability of these EWs was discussed, where the region of non-decomposable EWs of the first kind was partially determined and the decomposability of the EWs of the second kind was shown. Also, the EWs for which the feasible region was not a polygon and the problem was solved by approximate LP were discussed. Although, we considered only two kinds of Bell-states diagonal multispinor EWs manipulatable by exact or approximate LP, it is probable to define some other such multispinor EWs (even Bell-states non-diagonal multispinor ones) or some EWs with better approximations (may be solved by exact or approximate convex optimizations rather than LP ones) such that the region of PPT entangled states detectable by them be larger, where all of these cases are under investigation.

Appendix A

Throughout the paper, we have used the formalism of Euclidean Dirac fermions, i.e., the analytic continuation to imaginary time fermionic fields. In this continuation, the pseudo-orthogonal group $O(d-1, 1)$ is replaced with the orthogonal group $O(d)$, d being the Euclidean space dimension. Therefore Euclidean fermions transform under the spinorial representation of $O(d)$. In this appendix we define the algebra of Dirac γ matrices and exhibit matrices which realize the algebra in the Euclidean representation and explain our notations and conventions.

A.1 Dirac γ matrices

Space of even dimensions d . Let γ_μ , $\mu = 1, \dots, d$, be a set of d matrices satisfying the anticommuting relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} I, \quad (\text{A-i})$$

in which I is the identity matrix.

These matrices are the generators of a Clifford algebra similar to the algebra of operators acting on Grassmann algebras. It follows from relations (A-i) that the γ matrices generate an algebra which, as a vector space, has a dimension 2^d . In the following, we will give an inductive construction ($d \rightarrow d+2$) of hermitian matrices satisfying (A-i). In the algebra one element plays a special role, the product of all γ matrices. The matrix γ_S :

$$\gamma_S = i^{-d/2} \gamma_1 \gamma_2 \dots \gamma_{2n}, \quad (\text{A-ii})$$

anticommutes, because d is even, with all other γ matrices and $\gamma_S^2 = I$.

In calculations involving γ matrices, it is not always necessary to distinguish γ_S from other γ matrices. Identifying thus γ_S with γ_{d+1} , we have:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I, \quad i, j = 1, \dots, d, d+1. \quad (\text{A-iii})$$

The Greek letters $\mu \nu \dots$ are usually used to indicate that the value $d+1$ for the index has been excluded.

Space of odd dimensions. Equation (A-iii) shows that in odd dimensions, we can represent the γ matrices by taking the γ matrices of dimension $d-1$, to which we add γ_S . Note, however that in this case, in contrast to the even case, the γ matrices are not all algebraically independent.

A.2 An explicit construction of $\gamma_i^{(d)}$

It is sometimes useful to have an explicit realization of the algebra of γ matrices.

For $d = 2$, the standard Pauli matrices realize the algebra:

$$\begin{aligned}\gamma_1^{(d=2)} &\equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2^{(d=2)} \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \gamma_S^{(d=2)} &\equiv \gamma_3^{(d=2)} \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}\tag{A-iv}$$

The three matrices are hermitian, i.e., $\gamma_i = \gamma_i^\dagger$. The matrices γ_1 and γ_3 are symmetric and γ_2 is antisymmetric, i.e., $\gamma_1 = \gamma_1^t$, $\gamma_3 = \gamma_3^t$ and $\gamma_2 = -\gamma_2^t$.

To construct the γ matrices for higher even dimensions, we then proceed by induction, setting:

$$\begin{aligned}\gamma_i^{(d+2)} &= \sigma_1 \otimes \gamma_i^{(d)} = \begin{pmatrix} 0 & \gamma_i^{(d)} \\ \gamma_i^{(d)} & 0 \end{pmatrix}, \quad i = 1, \dots, d+1, \\ \gamma_{d+2} &= \sigma_2 \otimes I^{(d)} = \begin{pmatrix} 0 & -iI_d \\ iI_d & 0 \end{pmatrix},\end{aligned}\tag{A-v}$$

where, I_d is the unit matrix in $2^{d/2}$ dimensions.

As a consequence $\gamma_S^{(d+2)}$ has the form:

$$\gamma_S^{(d+2)} \equiv \gamma_{d+3}^{(d+2)} = \sigma_3 \otimes I_d = \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}.\tag{A-vi}$$

A straightforward calculation shows that if the matrices $\gamma_i^{(d)}$ satisfy relations (A-iii), the $\gamma_i^{(d+2)}$ matrices satisfy the same relations. By induction we see that the γ matrices are all hermitian. from (A-v), it is seen that, if $\gamma_i^{(d)}$ is symmetric or antisymmetric, $\gamma_i^{(d+2)}$ has the same property.

The matrix $\gamma_{d+2}^{(d+2)}$ is antisymmetric and $\gamma_S^{(d+2)}$ which is also $\gamma_{d+3}^{(d+2)}$ is symmetric. It follows immediately that, in this representation, all γ matrices with odd index are symmetric and all matrices with even index are antisymmetric, i.e.,

$$\gamma_i^t = (-1)^{i+1} \gamma_i. \quad (\text{A-vii})$$

Appendix B

In this appendix we prove the inequalities (3.19) and (3.30).

Proof of the inequalities (3.19):

In order to prove the inequalities (3.19), we first prove that the expectation value of the operator $I + \sum_{k=1}^{d+1} (-1)^{i_k} \underbrace{\gamma_k^{(d)} \otimes \dots \otimes \gamma_k^{(d)}}_m$ over an arbitrary pure product state $|\alpha_1\rangle|\alpha_2\rangle\dots|\alpha_m\rangle$ is non-negative.

By defining $b_i := \langle \psi^{(d)} | \gamma_i^{(d)} | \psi^{(d)} \rangle$, where $|\psi^{(d)}\rangle$ is an arbitrary pure state in the Hilbert space of dimension $2^{d/2}$, first we prove that $\sum_{i=1}^{2d+1} b_i^2 \leq 1$. We prove this by induction on d . First note that by using (A-v), the matrices $\gamma_i^{(d)}$ can be rewritten recursively as follows

$$\gamma_1^{(d)} = \gamma_1^{(d-2)} \otimes \sigma_1, \quad \gamma_2^{(d)} = \gamma_1^{(d-2)} \otimes \sigma_2, \quad \gamma_3^{(d)} = \gamma_1^{(d-2)} \otimes \sigma_3, \quad \gamma_i^{(d)} = \gamma_{i-2}^{(d-2)} \otimes I_2, \quad i = 4, \dots, d+1. \quad (\text{A-viii})$$

Now, we consider the pure state $|\psi^{(d)}\rangle$ as follows

$$|\psi^{(d)}\rangle = \beta_d |\psi^{(d-2)}\rangle | +x \rangle + \delta_d |\psi'^{(d-2)}\rangle | -x \rangle, \quad |\beta_d|^2 + |\delta_d|^2 = 1. \quad (\text{A-ix})$$

By using (A-viii), it is seen that by a rotation of magnitude $\pi/2$ about the x axes in the last component of $\gamma_i^{(d)}$, one can take the expectation values of $\gamma_2^{(d)}$ and $\gamma_3^{(d)}$ equal to zero, i.e., $b_2 = b_3 = 0$ (recall that $\gamma_1^{(d)} = \sigma_1 \otimes \dots \otimes \sigma_1$, $\gamma_2^{(d)} = \sigma_1 \otimes \dots \otimes \sigma_1 \otimes \sigma_2$ and $\gamma_3^{(d)} = \sigma_1 \otimes \dots \otimes \sigma_1 \otimes \sigma_3$). Therefore, we have

$$b_1 = \langle \psi^{(d)} | \gamma_1^{(d)} | \psi^{(d)} \rangle = |\alpha_d|^2 \langle \psi^{(d-2)} | \gamma_1^{(d-2)} | \psi^{(d-2)} \rangle - |\beta_d|^2 \langle \psi'^{(d-2)} | \gamma_1^{(d-2)} | \psi'^{(d-2)} \rangle, \\ b_i = \langle \psi^{(d)} | \gamma_i^{(d)} | \psi^{(d)} \rangle = |\alpha_d|^2 \langle \psi^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi^{(d-2)} \rangle + |\beta_d|^2 \langle \psi'^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi'^{(d-2)} \rangle, \quad i = 4, \dots, 2d+1. \quad (\text{A-x})$$

Then, we have

$$\begin{aligned}
\sum_i b_i^2 &= |\alpha_d|^4 \underbrace{(\langle \psi^{(d-2)} | \gamma_1^{(d-2)} | \psi^{(d-2)} \rangle^2 + \sum_{i=4}^{2d+1} \langle \psi^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi^{(d-2)} \rangle^2)}_{\leq 1} + \\
|\beta_d|^4 &\underbrace{(\langle \psi'^{(d-2)} | \gamma_1^{(d-2)} | \psi'^{(d-2)} \rangle^2 + \sum_{i=4}^{2d+1} \langle \psi'^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi'^{(d-2)} \rangle^2)}_{\leq 1} + 2|\alpha_d|^2 |\beta_d|^2 \{ \sum_{i=4}^{2d+1} \langle \psi^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi^{(d-2)} \rangle \times \\
&\langle \psi'^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi'^{(d-2)} \rangle - \langle \psi^{(d-2)} | \gamma_1^{(d-2)} | \psi^{(d-2)} \rangle \langle \psi'^{(d-2)} | \gamma_1^{(d-2)} | \psi'^{(d-2)} \rangle \} \leq \\
|\alpha_d|^4 + |\beta_d|^4 + 2|\alpha_d|^2 |\beta_d|^2 &\sqrt{\underbrace{\sum_{i=3}^{2d+1} \langle \psi^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi^{(d-2)} \rangle^2}_{\leq 1} \cdot \underbrace{\sum_{i=3}^{2d+1} \langle \psi'^{(d-2)} | \gamma_{i-2}^{(d-2)} | \psi'^{(d-2)} \rangle^2}_{\leq 1}} \leq \\
|\alpha_d|^4 + |\beta_d|^4 + 2|\alpha_d|^2 |\beta_d|^2 &= 1, \tag{A-xi}
\end{aligned}$$

where, we have used the hypothesis of induction in the first two inequalities and the Schwartz inequality in the third one.

Now, by using the fact that $|\langle \alpha_i | \gamma_k^{(d)} | \alpha_i \rangle| \leq 1$, $i = 1, 2, \dots, m$ we have

$$\begin{aligned}
Tr(\sum_{k=1}^{d+1} (-1)^{i_k} \underbrace{\gamma_k^{(d)} \otimes \dots \otimes \gamma_k^{(d)}}_m | \alpha_1 \rangle \langle \alpha_1 | \otimes \dots \otimes | \alpha_m \rangle \langle \alpha_m |) &\leq \\
\sum_{k=1}^{d+1} |\langle \alpha_1 | \gamma_k^{(d)} | \alpha_1 \rangle \langle \alpha_2 | \gamma_k^{(d)} | \alpha_2 \rangle \dots \langle \alpha_m | \gamma_k^{(d)} | \alpha_m \rangle| &\leq \\
\sum_{k=1}^{d+1} |\langle \alpha_1 | \gamma_k^{(d)} | \alpha_1 \rangle \langle \alpha_2 | \gamma_k^{(d)} | \alpha_2 \rangle| &\leq \sqrt{\underbrace{\sum_{k=1}^{d+1} (\langle \alpha_1 | \gamma_k^{(d)} | \alpha_1 \rangle)^2}_{\leq 1} \cdot \underbrace{\sum_{k=1}^{d+1} (\langle \alpha_2 | \gamma_k^{(d)} | \alpha_2 \rangle)^2}_{\leq 1}} \leq 1, \tag{A-xii}
\end{aligned}$$

where, we have used the Schwartz inequality in the third inequality and the fact that $\sum_{i=1}^{d+1} b_i^2 = \sum_{i=1}^{d+1} (\langle \psi^{(d)} | \gamma_i^{(d)} | \psi^{(d)} \rangle)^2 \leq 1$.

Therefore, the expectation value of the operator $I + \sum_{k=1}^{d+1} (-1)^{i_k} \underbrace{\gamma_k^{(d)} \otimes \dots \otimes \gamma_k^{(d)}}_m$ over any pure product state is non-negative, hence it is non-negative over any separable state ρ_s , since separable states can be written as convex combinations of pure product states.

Proof of the inequalities (3.30):

We consider the case $j = 1$; $i_1 = i_2 = i_3 = 0$, the proof of the other cases is similar. As regards the arguments of the proof of inequalities (3.19), it must be proved that the expectation value of the operator $I + A'_1 \otimes \dots \otimes A'_1 + A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1} + A'_{d+1} \otimes \dots \otimes A'_{d+1}$ over the pure product state $|\alpha_1\rangle \dots |\alpha_m\rangle$ is non-negative.

Now, by using the fact that $|\langle \alpha_i | A'_k | \alpha_i \rangle| \leq 1$, $i = 1, 2, \dots, m$ we have

$$\begin{aligned} & Tr\{(A'_1 \otimes \dots \otimes A'_1 + A'_{d/2+1} \otimes \dots \otimes A'_{d/2+1} + A'_{d+1} \otimes \dots \otimes A'_{d+1})|\alpha_1\rangle\langle\alpha_1| \otimes \dots \otimes |\alpha_m\rangle\langle\alpha_m|\} \leq \\ & |\langle\alpha_1|A'_1|\alpha_1\rangle \dots \langle\alpha_m|A'_1|\alpha_m\rangle| + |\langle\alpha_1|A'_{d/2+1}|\alpha_1\rangle \dots \langle\alpha_m|A'_{d/2+1}|\alpha_m\rangle| + |\langle\alpha_1|A'_{d+1}|\alpha_1\rangle \dots \langle\alpha_m|A'_{d+1}|\alpha_m\rangle| \leq \\ & |\langle\alpha_1|A'_1|\alpha_1\rangle| + |\langle\alpha_1|A'_{d/2+1}|\alpha_1\rangle| + |\langle\alpha_1|A'_{d+1}|\alpha_1\rangle| \leq 1, \end{aligned} \quad (\text{A-xiii})$$

where, we have used the fact that $A'_1 = -i\gamma_1^{(d)}\gamma_2^{(d)} = I \otimes \dots \otimes I \otimes \sigma_z$, $A'_{d/2+1} = \gamma_1^{(d)} = \sigma_x \otimes \dots \otimes \sigma_x$ and $A'_{d+1} = \gamma_2^{(d)} = \sigma_x \otimes \dots \otimes \sigma_x \otimes \sigma_y$ and so, by a rotation of magnitude $\pi/2$ about the z axis in the last component of A'_1 , $A'_{d/2+1}$ and A'_{d+1} , one can take the expectation values $|\langle\alpha_1|A'_{d/2+1}|\alpha_1\rangle|$ and $|\langle\alpha_1|A'_{d+1}|\alpha_1\rangle|$ equal to zero. \square

Appendix C

In this appendix, we show that the region of SSNNEV is convex if the feasible region be convex.

Let $W = a_0 I + \sum_i a_i O_i$ be a hermitian operator. Then, in order that W be an EW, the function $F(a, P)$ defined as

$$F(a, P) = a^T P + a_0 \quad (\text{A-xiv})$$

must be positive ($P_i := Tr(O_i \rho_s)$ for any separable state ρ_s), hence, the region of SSNNEV is defined by

$$\inf_P F(a, P) = \inf_P (a^T P + a_0) \geq 0. \quad (\text{A-xv})$$

Now, it must be proved that the region defined by (A-xv) is convex. To do so, note that $F(a, P)$ is an affine and therefore also linear function (recall that a function is affine if it is a sum of a linear function and a constant). Then, it is both convex and concave [45]. Now, we

recall the definition of the conjugate function and sublevel sets of a function as follows:

Definition 1 Let $f : R^n \rightarrow R$. The function $f^* : R^n \rightarrow R$ defined as

$$f^*(y) = \sup_{x \in \text{dom} f} (y^T x - f(x)), \quad (\text{A-xvi})$$

is called the conjugate of the function f (dom denotes the domain of f).

It is seen immediately that f^* is a convex function, since it is the pointwise supremum of a family of convex (indeed, affine) functions of y . This is true whether or not f is convex.

Definition 2 The α -sublevel set of a function $f : R^n \rightarrow R$ is defined as

$$C_\alpha = \{\alpha \in \text{dom} f | f(x) \leq \alpha\}. \quad (\text{A-xvii})$$

Sublevel sets of a convex function are convex, for any value of α [45].

Now, we consider the conjugate function of the constant function $f(P) = a_0$, for all P in the feasible region. Then, (A-xv) is equivalent to

$$\sup_P (-a^T P - a_0) \leq 0. \quad (\text{A-xviii})$$

By renaming $P' = -P$, (A-xviii) is written as

$$f^*(a) = \sup_{P'} (a^T P' - a_0) \leq 0. \quad (\text{A-xix})$$

It could be noticed that, the set $\{a \in \text{dom} f^* | f^*(a) \leq 0\}$ is the 0-sublevel set of the convex function f^* and so is a convex set. Therefore, we conclude that the set $\{a \in \text{dom} f^* | \inf_P (a^T P + a_0) \geq 0\}$ is convex. \square

It should be noticed that if the feasible region be a polygon, then the region of SSNNEV is also a polygon. Therefore, the apexes of the feasible region correspond to the hyperplanes surrounding the region of SSNNEV and vice versa, i.e., the feasible region and the region of SSNNEV are dual with each other.

Appendix D

Proof of the inequalities (6.110):

We prove the Eq.(6.110) only for the case $i_1 = \dots = i_{d+2} = 0$. The proof of the other cases is similar. Then, the Eq.(6.110) is given by

$$\sqrt{2} + \sum_{k=1}^{d+2} P_k \geq 0. \quad (\text{A-xx})$$

As before, it is sufficient to prove that the expectation value of the operator $\sqrt{2}I + \sum_{k=1}^{d+2} A_k \otimes \dots \otimes A_k$ with $A_{d+2} = \gamma_1^{(d)} \gamma_2^{(d)}$, over any pure product state $|\alpha_1\rangle \dots |\alpha_m\rangle$ is non-negative ($\sum_{k=1}^{d+2} P_k$ is the expectation value of the operator $\sum_{k=1}^{d+2} A_k \otimes \dots \otimes A_k$, over any separable state). To do so, we define $b_i = \langle \alpha_1 | A_i | \alpha_1 \rangle$, for $i = 1, \dots, d+2$ and evaluate the largest eigenvalue of $\sum_{k=1}^{d+2} b_k A_k$. Now, we note that

$$\left(\sum_{k=1}^{d+2} b_k A_k \right)^2 = \left(\sum_{k=1}^{d+1} b_k^2 + b_{d+2}^2 \right) I, \quad (\text{A-xxi})$$

where, we have used the fact that A_{d+2} anticommutes with A_1 and A_2 . Therefore, the eigenvalues of $(b_1 A_1 + b_2 A_2 \pm b_{d+2} A_{d+2})^2$ are given by

$$\lambda^2 = b_1^2 + b_2^2 + b_{d+2}^2 \leq 1 + \cos^2 2\theta \leq 2, \quad (\text{A-xxii})$$

where, we have used the fact that

$$b_{d+2} = \langle \alpha | A_{d+2} | \alpha \rangle = \langle \alpha | I \otimes \dots \otimes I \otimes \sigma_z | \alpha \rangle = \sum_{k=1}^{2^{d/2-1}} |\alpha_{2k-1}|^2 - \sum_{k=1}^{2^{d/2-1}} |\alpha_{2k}|^2 = 1 - 2 \sum_{k=1}^{2^{d/2-1}} |\alpha_{2k}|^2. \quad (\text{A-xxiii})$$

From the equality $\sum_{k=1}^{2^{d/2}} |\alpha_k|^2 = 1$, it can be seen that one can choose a parametrization for α_i such that $\sum_{k=1}^{2^{d/2-1}} |\alpha_{2k-1}|^2 = \cos^2 \theta$ and $\sum_{k=1}^{2^{d/2-1}} |\alpha_{2k}|^2 = \sin^2 \theta$. Then, (A-xxiii) will imply that $b_{d+2} = 1 - 2 \sin^2 \theta = \cos 2\theta$.

Proof of the inequalities (6.126):

We consider only the case of $i_1 = i_2 = i_3 = 0$ and $j = 1$. Then, the Eq.(6.126) is given by

$$P'_1 + P'_{d/2+1} - (-i)^m P'_{d+1} + P'_{3d/2+1} = P'_1 + P'_{d/2+1} \pm P'_{d+1} + P'_{3d/2+1} \leq 2, \quad (\text{A-xxiv})$$

Now, similar to the proof of Eq.(6.110) as in the above, we prove that the expectation value of the operator $2I + A'_1 \otimes A'_1 + A'_{d/2+1} \otimes A'_{d/2+1} \pm A'_{d+1} \otimes A'_{d+1} + A'_{3d/2+1} \otimes A'_{3d/2+1}$ over any pure

product state $|\alpha_1\rangle|\alpha_2\rangle$ is non-negative. By defining $b'_i = \langle\alpha_1|A'_i|\alpha_1\rangle$, for $i = 1, \frac{d}{2}+1, d+1, \frac{3d}{2}+1$, we need to evaluate the largest eigenvalue of $b'_1 A'_1 + b'_{d/2+1} A'_{d/2+1} \pm b'_{d+1} A'_{d+1} + b'_{3d/2+1} A'_{3d/2+1}$ as before. One can easily check that

$$(b'_1 A'_1 + b'_{d/2+1} A'_{d/2+1} \pm b'_{d+1} A'_{d+1} + b'_{3d/2+1} A'_{3d/2+1})^2 = \sum_i b_i'^2 I + 2b'_1 b'_{\frac{3d}{2}+1} A'_1 A'_{\frac{3d}{2}+1}, \quad (\text{A-xxv})$$

where, we have used the fact that $A'_{\frac{3d}{2}+1}$ anticommutes with $A'_{d/2+1}$ and A_{d+1} and commutes with A'_1 . Then, the eigenvalues of $(b'_1 A'_1 + b'_{d/2+1} A'_{d/2+1} \pm b'_{d+1} A'_{d+1} + b'_{3d/2+1} A'_{3d/2+1})^2$ are as follows

$$\begin{aligned} \lambda' = \sum_i b_i'^2 \pm 2b'_1 b'_{\frac{3d}{2}+1} &\leq 1 + b'_{\frac{3d}{2}+1} (b'_{\frac{3d}{2}+1} + 2b'_1) = 1 + \sin 2\phi (\cos^2 \theta \cos \theta' - \sin^2 \theta \cos \theta'') \times \\ &[2 \cos 2\theta + \sin 2\phi (\cos^2 \theta \cos \theta' - \sin^2 \theta \cos \theta'')] \leq 4, \end{aligned} \quad (\text{A-xxvi})$$

where, the maximum value 4 is obtained by taking $\phi = \pi/4, \theta = \theta' = 0$. Note that above, we have used the following equality

$$b'_{3d/2+1} = \langle\alpha|\gamma_3^{(d)}|\alpha\rangle = \langle\alpha|\sigma_x \otimes \dots \otimes \sigma_x \otimes \sigma_z|\alpha\rangle = 2\{Re(\sum_{k=1}^{2^{d/2}-2} \alpha_{2k-1}^* \alpha_{2^{d/2}-2k+1}) - Re(\sum_{k=1}^{2^{d/2}-2} \alpha_{2k}^* \alpha_{2^{d/2}-2k})\}. \quad (\text{A-xxvii})$$

Appendix E

Proof of the inequalities (4.50):

First we note that, by applying the transform $H \otimes I$ with $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$ on the first particle, the helicity basis (4.41) take the following form

$$|\psi_1\rangle = |00\rangle, \quad |\psi_2\rangle = |11\rangle \quad |\psi_3\rangle = |10\rangle \quad |\psi_4\rangle = |01\rangle \quad (\text{A-xxviii})$$

which are the same as Dirac's spinors. Also, this transformation changes the Bell-type states $|\Psi_i\rangle$, $i = 1, 2, \dots, 16$ to the traditional Bell states [8]-[11] which are obtained via the action of the Heisenberg group $H_{Z_2 \times Z_2} (\cong (Z_2 \times Z_2) \times (Z_2 \times Z_2) \rtimes (Z_2 \times Z_2))$ on the following maximally entangled state

$$|\Psi_{00}\rangle = \frac{1}{2} \sum_{i,j=0,1} |ij\rangle |ij\rangle, \quad (\text{A-xxix})$$

i.e., we have

$$|\Psi_{\mu\nu}\rangle = A_\mu \otimes A_\nu |\Psi_{00}\rangle = \sigma_\alpha \otimes \sigma_\beta \otimes \sigma_{\alpha'} \otimes \sigma_{\beta'} = \Omega^i S^j \otimes \Omega^k S^l \otimes \Omega^{i'} S^{j'} \otimes \Omega^{k'} S^{l'} |\Psi_{00}\rangle, \quad (\text{A-xxx})$$

where the operators $S = \sigma_x$ and $\Omega = \sigma_z$ known as shift and modulation operators are the generators of the Heisenberg group $H_{Z_2 \otimes Z_2}$. Then, it is sufficient to show that $|\Psi_{00}\rangle\langle\Psi_{00}|$ is written in terms of the diagonal elements $A_\mu \otimes A_\mu$. To do so, let

$$|\Psi_{00}\rangle\langle\Psi_{00}| = \sum_{\mu,\nu} b_{\mu\nu} A_\mu \otimes A_\nu. \quad (\text{A-xxxix})$$

where, $b_{\mu\nu} = \langle\Psi_{00}|A_\mu \otimes A_\nu|\Psi_{00}\rangle$. By taking $A_\mu = \sigma_\alpha \otimes \sigma_\beta$ and $A_\nu = \sigma_{\alpha'}^\dagger \otimes \sigma_{\beta'}^\dagger$ and using (A-xxix), we obtain

$$\begin{aligned} b_{\mu\nu} &= \sum_{i,i',j,j'} \langle i|\sigma_\alpha|i'\rangle \langle i|\sigma_{\alpha'}^\dagger|i'\rangle \langle j|\sigma_\beta|j'\rangle \langle j|\sigma_{\beta'}^\dagger|j'\rangle = \sum_{i,i',j,j'} \langle i|\Omega^k S^l|i'\rangle \langle i|S^{-l'}\Omega^{-k'}|i'\rangle \langle j|\Omega^r S^s|j'\rangle \langle j|S^{-s'}\Omega^{-r'}|j'\rangle \\ &= \sum_{i,i',j,j'} \langle i|\Omega^k|l+i'\rangle \langle i-l'|\Omega^{-k'}|i'\rangle \langle j|\Omega^r|s+j'\rangle \langle j-s'|\Omega^{-r'}|j'\rangle = \sum_{i,i',j,j'} \omega^{(k-k')i} \omega^{(r-r')j} \delta_{i,l+i'} \delta_{i-l',i'} \delta_{j,s+j'} \delta_{j-s',j'} \\ &= \delta_{kk'} \delta_{ll'} \delta_{rr'} \delta_{ss'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} = \delta_{\mu,\nu}, \quad \omega = e^{-\pi i} = -1. \end{aligned} \quad (\text{A-xxxix})$$

Proof for the fact that $W_{opt}(p)$ given in (4.83) is an entanglement witness

In order to show that $W_{opt}(p)$ in (4.83) is an EW, it must be proved that the expectation value of $W_{opt}(p)$ over any product state $|\gamma\rangle = |\alpha\rangle|\beta\rangle$ is non-negative. To do so, as it is seen from Eq.(4.70), we need to show that

$$\langle\gamma|\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p)|\gamma\rangle - \varepsilon(p) \geq 0. \quad (\text{A-xxxiii})$$

In order to prove (A-xxxiii), first we evaluate the minimum value of $\langle\gamma|\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0)|\gamma\rangle$ as follows

$$\langle\gamma|\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0)|\gamma\rangle = \frac{1}{120} \{ \langle\gamma|\gamma^0 \otimes \gamma^0|\gamma\rangle - \langle\gamma|\gamma^1 \otimes \gamma^1|\gamma\rangle - \langle\gamma|\gamma^2 \otimes \gamma^2|\gamma\rangle - \langle\gamma|\gamma^3 \otimes \gamma^3|\gamma\rangle - \langle\gamma|\gamma^5 \otimes \gamma^5|\gamma\rangle \} =$$

$$\frac{1}{120}\{b_0\langle\beta|\gamma^0|\beta\rangle - b_1\langle\beta|\gamma^1|\beta\rangle - b_2\langle\beta|\gamma^2|\beta\rangle - b_3\langle\beta|\gamma^3|\beta\rangle - b_5\langle\beta|\gamma^5|\beta\rangle\}, \quad (\text{A-xxxiv})$$

with $b_\mu := \langle\alpha|\gamma^\mu|\alpha\rangle$ for $\mu = 0, 1, 2, 3, 5$. By defining

$$O := b_0\gamma^0 - b_1\gamma^1 - b_2\gamma^2 - b_3\gamma^3 - b_5\gamma^5, \quad (\text{A-xxxv})$$

and using the fact that the eigenvalues of O are $\pm\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_5^2}$ (from the anti-commutativity of γ^μ , $\mu = 0, 1, 2, 3, 5$ we have $O^2 = (b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_5^2)I \otimes I$), we obtain

$$\langle\gamma|\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0)|\gamma\rangle = \frac{1}{120}\langle\beta|O|\beta\rangle \geq -\frac{1}{120}, \quad (\text{A-xxxvi})$$

where, we have used the fact that $\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_5^2} \leq 1$ (see the proof of the inequalities (3.19) given in appendix B). Therefore the minimum value of $\langle\gamma|\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0)|\gamma\rangle$ is equal to $-\frac{1}{120}$. Then, we can write

$$\begin{aligned} \langle\gamma|\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p)|\gamma\rangle &= \frac{1}{\cosh^2(\xi)}\langle\gamma|(D \otimes D)(\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0))(D^\dagger \otimes D^\dagger)|\gamma\rangle = \\ &= \frac{1}{\cosh^2(\xi)}\langle\gamma'|(\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0))|\gamma'\rangle \geq -\frac{1}{120 \cosh^2(\xi)}, \end{aligned} \quad (\text{A-xxxvii})$$

where, $|\gamma'\rangle := (D^\dagger \otimes D^\dagger)|\gamma\rangle$ is another product state and so the expectation value of $\rho_s(0) - \rho_{ent}^{(1,0,0,0)}(0)$ over it is larger than $-\frac{1}{120}$. Therefore, by using (4.82) and (A-xxxvii), one can obtain

$$\begin{aligned} \langle\gamma|\rho_s(p) - \rho_{ent}^{(1,0,0,0)}(p)|\gamma\rangle - \varepsilon(p) &\geq \frac{1}{120 \cosh^2(\xi)} \left\{ \frac{\cosh^8(\xi/2)}{5 \cosh^4(\xi)} [5(1 + \tanh^8(\xi/2)) + \right. \\ &\quad \left. 28(\tanh^2(\xi/2) + \tanh^6(\xi/2)) + 126 \tanh^4(\xi/2)] - 1 \right\} \geq 0. \end{aligned} \quad (\text{A-xxxviii})$$

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